RESEARCH STATEMENT

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Broadly speaking, I study functional analysis, in particular operator theory and operator algebras. My research spans several different particular fields, and indeed I love to jump between different problems that require different tools. In this document I will discuss some highlights of my work.

The research I have done splits into several categories by topic. In each section below, I list my publications related to the topic, and then I discuss a sample of my achievements. I state some results as theorems, but the formulations are sometimes a bit simplified for the sake of this presentation; please see the papers for details.

The order of the topics has some chronological justification, but since I worked on different projects at several different times, the report does not necessarily proceed from older to newer results. I hope that the readers will choose the section which is closest to their expertise and interest, read that first, and then proceed to other subjects that interest them.

1. Noncommutative dynamics and dilation theory [40, 55, 56, 58, 60, 64, 65, 66]

Noncommutative dynamics, also know as *quantum dynamics*, is the mathematical formalism concerning time evolution in a quantum mechanical system. The central object of study is a semigroup $\varphi = \{\varphi_t\}_{t\geq 0}$ of completely positive maps (for short: *CP-semigroup*) on $B(H)$, where $B(H)$ is the algebra of all bounded operators on some Hilbert space H. When each φ_t is a $*$-endomorphism of $B(H)$, then we say that φ is an *E-semigroup*.

The connection between quantum dynamics and dilation theory was made in the '70s by Evans and Lewis [22]. They showed that if φ is a *uniformly continuous* CP-semigroup, then it can be dilated to an E-semigroup in the following sense: there is some Hilbert space K containing H and an E-semigroup $\alpha = \{\alpha_t\}_{t\geq 0}$ on $B(K)$ such that

$$\varphi_t(T) = P_H \alpha_t(T) P_H, \tag{1}$$

for all $T \in B(H)$ and $t \in \mathbb{R}_+$, where P_H is the orthogonal projection of K onto H.

However, in general a CP-semigroup is not uniformly continuous (since this would imply that the Hamiltonian is bounded!). The only continuity which one may naturally assume is the following:

$$\lim_{t\to t_0} \langle \varphi_t(T)h, g\rangle = \langle \varphi_{t_0}(T)h, g\rangle , \tag{2}$$

for all $T \in B(H)$, $h, g \in H$ and $t \in \mathbb{R}_+$. It was only in the early '90s when Bhat proved the following theorem, which is one of the cornerstones of noncommutative dynamics.

Theorem (Bhat's Dilation Theorem [7]). *Every CP-semigroup, continuous in the sense of (2), has an E-dilation; that is, there is an E-semigroup α, likewise continuous, which satisfies (1).*

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My main contribution to this field is the extension of Bhat's Theorem to CP-semigroups $\varphi = \{\varphi_s\}_{s\in\mathcal{S}}$ which are parameterized by some semigroup $\mathcal{S}$, rather then by the semigroup $\mathbb{R}_+$. The cases of greatest interest are $\mathcal{S} = \mathbb{N}^k$, which corresponds to k commuting CP maps, or $\mathcal{S} = \mathbb{R}_+^k$, which corresponds to k commuting one-parameter CP-semigroups. My first main result is that every pair of *strongly commuting* CP-semigroups has an E-dilation.

Theorem 1 (Shalit [55, 60])**.** *Every strongly commuting two parameter CP-semigroup has an E-dilation. If the CP-semigroup preserves the unit, then so does the dilation.*

First I established the unital case [55] (see also [56] for a further analysis), and later the nonunital case [60]. In order to obtain these results I proved several new results regarding the existence of *isometric dilations* for representations of *product systems* [55, 58, 60].

My second main group of results is the development of a framework for studying CP-semigroups and dilations over general semigroups [66]. Together with my PhD supervisor Baruch Solel, we introduced and studied the notion of a *subproduct system*, which is a generalization of the notion of a *product system*, introduced by Arveson [4]. A subproduct system is a family $X = \{X(s)\}_{s\in\mathcal{S}}$ of Hilbert spaces that satisfies

$$X(s+t) \subseteq X(s) \otimes X(t) \ , \quad \text{for all} \quad s,t \in \mathcal{S}.$$

To every CP-semigroup one can attach a subproduct system in a canonical way. Roughly, we proved the following theorem.

Theorem 2 (Shalit-Solel [66])**.** *There is a bijective correspondence between CP-semigroups, on the one hand, and subproduct systems and subproduct system representations, on the other hand. In this correspondence, E-dilations corespond to isometric dilations of subproduct system representations combined with embeddings of subproduct systems into product systems.*

The above bijective correspondence allows us to use subproduct systems and their representations as the main technical tool for either constructing dilations, or for showing that they do not exist. We showed that when the CP-semigroup is *unital*, a necessary and sufficient condition for the existence of an E-dilation is that the associated subproduct system may be embedded into a *product system*. This result contains all known Bhat-type theorems for unital semigroups (at least the algebraic part), and produces some new ones as well. Further, we were able to construct three commuting CP maps which have no E-dilation. This result has been expected for more then a decade. In fact, we also obtained the following surprising result

Theorem 3 (Shalit-Solel [66])**.** *There exists a subproduct system over the semigroup* $\mathbb{N}^3$ *that cannot be embedded into a product system. Consequently, there exist three commuting CP maps* $\theta_1, \theta_2, \theta_3$*, such that for all* $\lambda \in (0,1)$*, the triple* $\lambda\theta_1, \lambda\theta_2, \lambda\theta_3$ *has no E-dilation.*

In later work, using this framework, Michael Skeide and I managed to show that there exist three commuting *unital* CP maps for which there is no E-dilation [64]. This result was rather unexpected (see the introduction of that paper for and explanation why Theorems 3 and 4 are somewhat surprising), and shows how the "quantum" dilation theory exhibits phenomena that have no counterpart in classical dilation theory.

Theorem 4 (Shalit-Skeide [64])**.** *There exists three commuting* unital *CP maps* $\theta_1, \theta_2, \theta_3$ *that have no E-dilation.*

In the above discussion, the algebra $B(H)$ can be replaced by a von Neumann algebra $\mathcal{M}$; the dilated E-semigroup then must act on a von Neumann algebra $\mathcal{R}$ of the same type as $\mathcal{M}$. The subproduct system then becomes something somewhat more complicated: the family $X = \{X(s)\}_{s\in\mathcal{S}}$ now consists not of Hilbert spaces, but of *Hilbert W*-modules*.

If one wishes to extend the theory to CP-semigroups that act on a C*-algebra, then the machinery that I developed with Solel breaks down. One needs then subproduct systems of *C*-corresepondences*, and there are new topological and algebraic niceties. During the past several years I have been working with Michael Skeide on developing the definitive theory [65]. We obtain the following theorem.

Theorem 5 (Shalit-Skeide [65])**.** *To every semigroup $\varphi = \{\varphi_s\}_{s\in\mathcal{S}}$ of completely positive maps on a C*-algebra $\mathcal{B}$, there corresponds a subproduct system of Hilbert $\mathcal{B}$ correspondences $\{E_s\}_{s\in\mathcal{S}}$, and a unital unit $\{\xi_s\}_{s\in\mathcal{S}}$ that satisfies $\xi_{s+t} = \xi_s \otimes \xi_t$ (in the identification $E_{s+t} \subseteq E_s \otimes E_t$) such that*

$$\varphi_s(b) = \langle \xi_s, b\xi_s\rangle \quad , \quad b \in \mathcal{B}, s \in \mathcal{S}.$$

The semigroup φ has a unital E-dilation of specified kind, if and only if the subproduct systems embeds into a product system.

If one wishes to treat all dilations, and not only those of the specified kind, then the situation is more complex. For example, if we just know that there exists *some* dilation, then the subproduct system embeds into a *superproduct system*.

Our work sheds light on all previous results. In particular, we recover the result of Bhat and Solel that every pair of commuting CP maps on a von Neumann algebra has an E-dilation, and we strengthen it by showing that if the original maps are unital then then so is the dilation.

Theorem 6 (Shalit-Skeide [65])**.** *For every pair of commuting unital CP maps θ_1, θ_2 on a von Neumann algebra $\mathcal{M}$, the corresponding subproduct system embeds into a product system. Consequently, every such pair has a unital E-dilation.*

We also revisit my first result on this subject — the existence of an E-dilation for a pair of *strongly commuting* CP-semigroups — to clarify why strong commutation had to arise, and to provide a large class of pairs of commuting CP-semigroups that satisfy this assumption.

In [67], my former master's student Alex Vernik, using the tools that I developed, obtained the following interesting result (this was inspired by a similar result for isometric dilations of contractions due to David Opela, but the proof is significantly more difficult in the case of CCP maps).

Theorem (Vernik [67]**).** *Let $\theta_1, \ldots, \ldots \theta_k$ be CCP maps on a von Neumann algebra $\mathcal{M}$, that commute accoding to a graph G (that is, there is a graph G on the vertices $1, \ldots, k$, such that θ_i commutes with θ_j if and only if there is an edge in G connecting i and j). If G has no cycles, then there is dilation consisting of *-endomorphisms that commute according to G. Conversely, if G is a finite graph that has a cycle, then there are CCP maps that commute according to G but, have no E-dilation commuting according to G.*

Subproduct systems and superproduct systems turn out to be fundamental. Besides being relevant to dilation theory, subproduct systems are useful for other aspects in non-commutative dynamics (for example, Bhat-Mukherjee used them to compute the index of one-parameter CP semigroups). Superproduct systems have also appeared in the work of

Margetts and Srinivasan on semigroups of *-endomorphisms on type II von Neumann algebras. Moreover, subproduct system give a unified framework within which to study a very wide class of operator algebras, as I will explain in Section 3.

2. Continuity properties of one-parameter semigroups of operators [37, 40, 60]

As mentioned in the previous section, Bhat's Theorem is a cornerstone in noncommutative dynamics. In fact, there are at least five different published proofs. Most researchers, when referring to Bhat's Theorem, refer to the version that I stated above. The funny thing is that this version was never really fully proved. To go through, all of the proofs that I am aware of need the following crucial fact: *if $\varphi = \{\varphi_t\}_{t\geq 0}$ is a CP-semigroup which is continuous in the sense of (2), then φ satisfies the (seemingly much stronger) continuity condition*

$$\lim_{t\to t_0} \varphi_t(T)h = \varphi_{t_0}(T)h, \tag{3}$$

for all $h \in H$, $T \in B(H)$ and $t_0 \geq 0$. Some authors overlooked this problem and some relied on incomplete proofs for this subtle assertion. Note that the continuity in (2) is weaker than the weakest continuity one ever encounters in the theory of one-parameter semigroups on Banach spaces, thus one cannot use the Hille-Phillips-Yosida theory. By going deeply into all the proofs, I found this gap in the literature, and together with the collaboration of Daniel Markiewicz we proved in [40] that continuity in the sense of (2) implies continuity in the sense of (3).

Theorem 7 (Markiewicz-Shalit [40])**.** *A one parameter semigroup of completely positive maps on a von Neumann algebra $\mathcal{M}$ is point-weak continuous, if and only if it is point-strong continuous.*

Besides filling the gap in the literature, I needed this result for my work on two-parameter semigroups. For the construction of the dilation in [60], I also required a theorem that allows to extend a densely parameterized semigroup of CP maps to a one-parameter semigroup (see [60, Theorem 5.2]). This led me tackle, jointly with Eliahu Levy, a similar question regarding one-parameter semigroups on Banach spaces.

Theorem 8 (Levy-Shalit [37])**.** *Let $\mathcal{S} \subseteq \mathbb{R}_+$ be a dense subsemigroup (e.g., $\mathcal{S} = \mathbb{Q}$). Let $T = \{T_s\}_{s\in\mathcal{S}}$ be a family of contractions on a separable, reflexive Banach space X, which forms a weakly continuous semigroup, in the sense that*

(1) $T_0 = I$,
(2) $T_s \circ T_t = T_{s+t}$,
(3) $\lim_{\mathcal{S}\ni s\to s_0} y(T_s(x)) = y(T_{s_0}(x))$ *for all* $x \in X, y \in X^*$ *and* $s_0 \in \mathcal{S}$.

Then $T = \{T_s\}_{s\in\mathcal{S}}$ extends to a weakly (and hence, strongly) continuous semigroup $T = \{T_t\}_{t\in\mathbb{R}_+}$ on X.

Dear reader: if you think (like many mistakenly have) that this must be a very easy theorem, then please hold that thought for a moment while you write down the proof. Ah. It is worth mentioning that while we do not know whether this result is optimal, we have an example in our paper that one cannot drop all our assumptions.

3. Noncommutative multivariable operator theory and operator algebras [13, 32, 66]

3.1. **Subproduct systems.** Given a subproduct system $X = \{X(n)\}_{n\in\mathbb{N}}$, one can construct out of it a number of operator algebras; most interestingly, the *tensor algebra* $\mathcal{T}_+(X)$, the *Toeplitz algebra* $\mathcal{T}(X)$, and the *Cuntz algebra* $\mathcal{O}(X)$. They are defined as follows.

Let

$$\mathcal{F}(X) = \bigoplus_{n=0}^{\infty} X(n)$$

be the *Fock space* over X. The tensor algebra $\mathcal{T}_+(X)$ is the norm closed algebra generated by the operators S_ξ, $\xi \in X(m)$, given by the shift operators

$$S_\xi(\eta) = P_{X(m+n)}(\xi \otimes \eta) \quad , \quad \eta \in X(n).$$

Of course $P_{X(m+n)}$ denotes the projection of $X(m) \otimes X(n)$ onto $X(m+n)$. The Toeplitz algebra is given by $C^*(\mathcal{T}_+(X))$ and the Cuntz algebra is a quotient of $C^*(\mathcal{T}_+(X))$, typically by the compacts : $\mathcal{O}(X) =_{typically} C^*(\mathcal{T}_+(X))/\mathcal{K}(\mathcal{F}(X))$.

When the subproduct system has the simple form $X(n) = E^{\otimes n}$ for some fixed C*-correspondence E, then these algebras are the *tensor algebra* $\mathcal{T}_E^+$ of Muhly-Solel [43], and the *Toeplitz-Pimsner algebras* $\mathcal{T}_E$ and *Cuntz-Pimsner algebras* $\mathcal{O}_E$ of Pimsner and Katsura [48, 33]. This class contains crossed product algebras, graph algebras, the Cuntz algebras, and more. By letting X range over all subproduct systems, one gets a by-far richer class, including also algebras of analytic functions on balls or on analytic varieties, that do not arise in the case $X(n) = E^{\otimes n}$.

In [66, 13, 32] (in this order) I worked with collaborators on the problem of classifying the nonselfadjoint algebras $\mathcal{T}_+(X)$ for several interesting classes of subproduct systems. We also studied related C*-algebras, and, in particular, we identified the C*-envelope of $\mathcal{T}_+(X)$.

In [32], Kakariadis (who was my postdoc) and I studied the case where X corresponds to a monomial ideal. This generalizes the case where X is a subproduct system associated with a subshift, a connection that was introduced in [66].

Every monomial ideal $\mathcal{I}$ in the free algebra $\mathbb{C}\langle z_1, \ldots, z_d\rangle$ corresponds to a subproduct system of Hilbert spaces $X = \{X(n)\}_{n\in\mathbb{N}}$. In our work, Kakariadis and I introduced the "minimal" C*-correspondence E constructed from the shift operators, and we sorted out the relationship between the various algebras.

Theorem 9 (Kakariadis-Shalit [32])**.** *The algebra $\mathcal{T}(X)$ is a quotient of $\mathcal{T}_E$. It is isomorphic to $\mathcal{T}_E$ if and only if the ideal is trivial. When the ideal is not trivial, $\mathcal{O}_E \cong \mathcal{T}(X)$ if and only if the correspondence E is faithful, and otherwise $\mathcal{O}_E \cong \mathcal{T}(X)/\mathcal{K}(\mathcal{F}(X))$.*

Among many other things, we classified completely the operator algebras $\mathcal{T}_+(X)$ up to isomorphism:

Theorem 10 (Kakariadis-Shalit [32])**.** *Let X and Y be two subproduct systems corresponding to monomial ideals $\mathcal{I}$ and $\mathcal{J}$. The following are equivalent:*

(1) *$\mathcal{T}_+(X)$ is algebraically isomorphic to $\mathcal{T}_+(Y)$.*
(2) *$\mathcal{T}_+(X)$ is completely isometrically isomorphic to $\mathcal{T}_+(Y)$.*
(3) *$\mathcal{I} = \mathcal{J}$, up to a permutation of the variables.*

We also identified the C*-envelopes, and we found the following dichotomy.

Theorem 11 (Kakariadis-Shalit [32])**.** *The C*-envelope $C^*_{env}(\mathcal{T}_+(X))$ of the tensor algebra $\mathcal{T}_+(X)$ is equal to either $\mathcal{T}(X)$ or $\mathcal{O}(X)$ (and we have a complete characterization of when this happens).*

This dichotomy was a part of my program to understand the behavior of C*-envelopes of tensor algebras of subproduct systems. Previously, we showed in [13] that if X is a commutative subproduct system, then $C^*_{env}(\mathcal{T}_+(X)) = \mathcal{T}(X)$. It was also known that if $X(n) = E^{\otimes n}$ is a product system, then $C^*_{env}(\mathcal{T}_+(X)) = \mathcal{O}(X)$. We were curious to know what controls where the C*-envelope decides to fall between $\mathcal{T}(X)$ and $\mathcal{O}(X)$. In recent work of Dor-On (who was previously my master's student) and Markiewicz, the C*-envelopes of a class of subproduct systems was studied, and it was shown that it possible for C*-envelopes of subproduct systems to fall strictly between $\mathcal{T}(X)$ and $\mathcal{O}(X)$.

Finally, in my work with Kakariadis we defined a multivariable partial dynamical system (Ω, α), that encodes the language given by the ideal, and we show that this dynamical system completely classifies the tensor algebras $\mathcal{T}^+_E$.

Some of the work done in [13] will be detailed further in Section 4.1 below.

3.2. **Noncommutative function theory.** In a recent work [52], we studided algebras of bounded noncommutative (nc) anlaytic functions on the nc unit ball. We identify the algebras of bounded analytic functions on nc subvarieties of the unit ball as multiplier algebras of certain nc reproducing kernel Hilbert spaces, we show that these spaces have the complete Pick property, and we classify these spaces. One of our main results is the following:

Theorem 12 (Salomon-Shalit-Shamovich [52])**.** *Let $\mathfrak{V} \subseteq \mathfrak{B}_d$ and $\mathfrak{W} \subseteq \mathfrak{B}_e$ be nc varieties. Then $H^\infty(\mathfrak{V})$ and $H^\infty(\mathfrak{W})$ are completely isometrically isomorphic via a weak-$*$ continuous map, if and only if $\mathfrak{V}$ and $\mathfrak{W}$ are biholomorphically equivalent, in the sense that there exists a nc holomorphic map $G : \mathfrak{B}_e \to \mathfrak{B}_d$ and a nc holomorphic map $H : \mathfrak{B}_d \to \mathfrak{B}_e$ such that $G\big|_{\mathfrak{W}} = (H\big|_{\mathfrak{V}})^{-1}$.*

We show that weak-* continuity is automatic when the number of variables is finite. This investigation sheds light on results discussed in Sections 3.1 and 4.1.

4. Commutative multivariable operator theory and operator algebras [8, 11, 13, 14, 34, 35, 36, 44, 42, 59]

Fix some integer d, and let $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_d]$ be the algebra of complex polynomials in d variables. For a d-tuple $\alpha = (\alpha_1, \ldots, \alpha_d)$, we write

$$z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}.$$

On $\mathbb{C}[z]$ we define a norm as follows:

$$\left\| \sum c_\alpha z^\alpha \right\|^2 = \sum |c_\alpha|^2 \frac{\alpha_1! \cdots \alpha_d!}{(\alpha_1 + \ldots + \alpha_d)!}.$$

This norm is induced by an inner product, and the completion of $\mathbb{C}[z]$ by this norm is then a Hilbert space, called the *Drury-Arveson space*, and it is denoted H^2_d. It turns out that H^2_d is a *reproducing kernel Hilbert space* of analytic functions on the unit ball $\mathbb{B}_d \subset \mathbb{C}^d$ (see my survey [63] for an overview).

On H^2_d we define a d-tuple of operators $(S_1, \ldots, S_d)$, called the d-shift, as follows. For $f \in H^2_d$ and $j = 1, \ldots, d$, we define $S_j f$ to be the function

$$(S_j f)(z) = z_j f(z).$$

$(S_1, \ldots, S_d)$ is seen to be a *d-contraction*, which means that S_i commutes with S_j and that $(S_1, \ldots, S_d)$, when considered as a row operator from $H^2_d \oplus \ldots \oplus H^2_d$ to H^2_d, is a contraction. The d-shift is important because (as Drury [19], Arveson [5] and Popescu [49] proved) it is a *universal model* for all d-contractions. It is also the universal complete Pick space [1].

4.1. **The isomorphism problem for complete Pick algebras** [11, 13, 14, 36, 42, 51]**.** Let I be a homogeneous radical ideal in $\mathbb{C}[z]$, and let $M = I^\perp$ be the orthogonal complement of I in H^2_d. Let $(S^I_1, \ldots, S^I_d)$ denote the compression of the d-shift onto M. It is known [12, 50] that the unital norm closed algebra generated by $S^I = (S^I_1, \ldots, S^I_d)$, which we denote $\mathcal{A}_I$, is the *universal operator algebra* generated by a d-contraction $(T_1, \ldots, T_d)$ that satisfies

$$p(T_1, \ldots, T_d) = 0 \quad , \quad p \in I.$$

A natural question is: to what extent does I determine the structure of $\mathcal{A}_I$? The answer is as follows: *Let $V(I)$ be the affine algebraic variety associated with I. Then the geometry of $V(I)$ determines the structure of $\mathcal{A}_I$.* Different interpretations of geometry correspond to different interpretations of structure. It is not hard to see that $V(I)$ and $V(J)$ are isomorphic as algebraic varieties if, and only if, the (not closed) algebras $\mathrm{alg}(S^I_1, \ldots, S^I_d)$ and $\mathrm{alg}(S^J_1, \ldots, S^J_d)$ are isomorphic as algebras. The analytic problem gives a finer classification.

Theorem 13 (Davdison-Ramsey-Shalit [13], Hartz [28])**.** *With the notation above:*

1. *$\mathcal{A}_I$ and $\mathcal{A}_J$ are isometrically isomorphic if and only if they are unitarily equivalent, and this happens if and only if there is a unitary map sending $V(I)$ onto $V(J)$.*
2. *$\mathcal{A}_I$ and $\mathcal{A}_J$ are algebraically isomorphic if, and only if, there is an invertible linear map T on $\mathbb{C}^d$ that sends $V(I)$ onto $V(J)$ and is length preserving on $V(I)$.*

Hartz also deserves significant credit for the above theorem, since the original result from [13] required additional hypotheses, which were removed by Hartz in [28] (see also [29] for additional significant contributions by Hartz to the isomorphism problem).

The condition *there is an invertible linear map T on $\mathbb{C}^d$ that sends $V(I)$ onto $V(J)$ and is length preserving on $V(I)$* is rather rigid. We proved that if $V(I)$ is a nonlinear hyper-surface, or if it is irreducible, then T must be a unitary (but there *are* cases when it is not). From this follows an operator-algebraic rigidity result:

Theorem 14 (Davdison-Ramsey-Shalit [13])**.** *Let I be a prime or a principal ideal. If $\mathcal{A}_I$ and $\mathcal{A}_J$ are algebraically isomorphic, then they are isometrically isomorphic, and in this case they are in fact unitarily equivalent.*

The work with Davidson and Ramsey has naturally led us to study the case of nonhomogeneous ideals. However, to make progress, we needed to change the point focus from ideals to varieties. A rather comprehensive overview of my work on this problem can be found in the survey paper [51].

The setting of the modified problem is, roughly, as follows. Fix $d \in \mathbb{N}$ ($d = \infty$ is also treated, but omitted from this overview). Let $\mathcal{M}_d$ be the multiplier algbera of H^2_d. Thus,

$$\mathcal{M}_d = \{f : \mathbb{B}_d \to \mathbb{C} : fh \in H^2_d \text{ for all } h \in H^2_d\} \subseteq H^\infty(\mathbb{B}_d).$$

For an analytic subvariety $V \subseteq \mathbb{B}_d$, we denote

$$\mathcal{M}_V = \{f\big|_V : f \in \mathcal{M}_d\}.$$

Strictly speaking, we are only interested in *multiplier varieties*, not in arbitrary holomorphic varieties. A multiplier variety is the joint zero set of multipliers.

When V is a homogeneous variety, then $\mathcal{M}_V$ is simply the closure in the weak operator topology of the algebra $\mathcal{A}_{I(V)}$ discussed above (where $I(V)$ is the ideal of polynomials vanishing on V), and it is possible to transfer results from the norm closed algebra $\mathcal{A}_{I(V)}$ to the weak-operator closed algebra $\mathcal{M}_V$, and back. For general varieties, it turned out that the weak-operator closed algebras are more accessible ([14, Section 7] contains further discussion). Regarding the isomorphism problem for the algebras $\mathcal{M}_V$, we have the following theorem.

Theorem 15 (Davdison-Ramsey-Shalit [14])**.** *Let $V, W \subseteq \mathbb{B}_d$ be two multiplier varieties. Then $\mathcal{M}_V$ and $\mathcal{M}_W$ are isometrically isomorphic, if and only if there is a conformal automorphism of the ball that maps V onto W.*

With regards to algebraic isomorphism, we have the following direction.

Theorem 16 (Davdison-Ramsey-Shalit [14])**.** *Let $V, W \subseteq \mathbb{B}_d$ be two multiplier varieties that are a union of a discrete variety and a finite union of irreducible varieties. If $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic, then V and W are biholomorphic.*

We have a counter example in [11] showing that the converse does not hold in general. However, we do have some partial converses. The following theorem is an improvement to earlier results by Alpay-Putinar-Vinnikov and Arcozzi-Rochberg-Sawyer.

Theorem 17 (Kerr-McCarthy-Shalit [36])**.** *Let $V \subseteq \mathbb{B}_d$ be the image of a finite Riemann surface by a proper holomorphic map that has finitely many ramification points, and extends to an injective and regular C^2 function on the boundary. Then $\mathcal{M}_V = H^\infty(V)$, with comparable norms. It follows that if W is another such variety then $\mathcal{M}_V$ is isomorphic to $\mathcal{M}_W$ if and only if V and W are biholomorphic.*

Early versions required that V meet the boundary of the ball transversally, but in [11] we proved a Hopf-type-lemma that shows that one dimensional analytic varieties as above always meet the boundary transversally.

As a striking corollary to the above theorem, we obtain a Henkin type extension result.

Theorem 18 (Kerr-McCarthy-Shalit [36])**.** *Let $V \subseteq \mathbb{B}_d$ be a variety as in the previous theorem. Then there exists a constant $C > 0$, such that for every $f \in H^\infty(V)$, there exists a multiplier $F \in \mathcal{M}_d$ such that $F\big|_V = f$ and $\|F\|_{mult} \leq C\|f\|_\infty$ (and, in particular $F \in H^\infty(\mathbb{B}_d)$ and $\|F\|_\infty \leq C\|f\|_\infty$).*

Finally, in [42] John McCarthy and I studied Hilbert function spaces of Dirichlet series. We characterized when such a space has the complete Pick property, and then we discovered that many of the spaces that have the complete Pick property, are actually *weakly isomorphic* to the Drury-Arveson space H^2_d for some d. This means, in particular, that their multiplier algebras are universal. As a sampler, we state the following theorem.

Theorem 19 (M^cCarthy-Shalit [42])**.** *Let* $P(s) = \sum_p p^{-s}$ *denote the prime zeta function, and define the kernel*

$$k(s,u) = \frac{P(2)}{P(2-s-\overline{u})}.$$

Let $\mathcal{H} = \mathcal{H}(k)$ *be the corresponding reproducing kernel Hilbert space on the half plane* $\{z : \Re z > 0\}$*. Then its multiplier algebra* $\mathrm{Mult}(\mathcal{H})$ *is completely isometrically isomorphic to* $\mathcal{M}_\infty = \mathrm{Mult}(H^2_\infty)$*. In particular, every complete Pick algebra is a quotient of* $\mathrm{Mult}(\mathcal{H})$*.*

4.2. **Essential normality and Arveson's conjecture** [8, 34, 35, 59]**.** Each operator S_i commutes with all operators S_j, but does *not* commute with S_j^* for any j. However, the commutator $[S_i, S_j^*] := S_i S_j^* - S_j^* S_i$ is a compact operator for all i, j. A d-contraction $(T_1, \ldots, T_d)$ is said to be *essentially normal* if $[T_i, T_j^*]$ is compact for all i, j. A subspace $M \subseteq H^2_d$ is said to be *essentially normal* if the compression of $(S_1, \ldots, S_d)$ to M is essentially normal. Arveson conjectured that for any homogeneous ideal $I \subseteq \mathbb{C}[z]$, the closure of I in H^2_d is essentially normal. There is a refined conjecture due to Douglas, that says that the quotient of H^2_d by such an ideal is p-essentially normal for all $p > \dim I$, meaning that $|[T_i, T_j^*]|^p$ is a trace class operator. For brevity I will not give details on my results on p-essential normality.

Arveson's conjecture has been shown, early on, to hold true in some special cases, see, e.g., [6, 15, 27]. In recent years, some new techniques from harmonic analysis have been introduced [18, 23, 24], and probably the best result as of now is the validity of the conjecture for the smooth case [17, 21].

My first main contribution to this problem is the introduction of algorithmic techniques from computational algebraic geometry into the picture. I introduced the following notion.

Definition 20. *An ideal* $I \triangleleft \mathbb{C}[z]$ *is said to have the* approximate stable division property *if there are elements* $f_1, \ldots, f_k \in I$ *and a constant* A *such that for every* $\epsilon > 0$ *and for every* $h \in M$*, one can find polynomials* $g_1, \ldots, g_k \in \mathbb{C}[z]$ *such that*

$$\|h - \sum_{i=1}^{k} g_i f_i\| \leq \epsilon, \tag{4}$$

together with the norm constraint

$$\sum_{i=1}^{k} \|g_i f_i\| \leq A\|h\|. \tag{5}$$

If ϵ *can be chosen* 0*, then we say that* I *has the* stable division property*. The set* $\{f_1, \ldots, f_k\}$ *is said to be an* (approximate) stable generating set*.*

In [59] I studied the homogeneous case, and in [8] we studied the nonhomogeneous case.

Theorem 21 (Shalit, Biswas-Shalit [8, 59])**.** *If an ideal* $I \triangleleft \mathbb{C}[z]$ *has the approximate stable division property then the subspace* $M = \overline{I} \subseteq H^2_d$ *is essentially normal.*

There is also a vector valued version of the above result, and it works in a large family of Hilbert modules, not just H^2_d. Proving the above theorem would have been pointless without finding examples that satisfy the condition.

Theorem 22 (Shalit, Biswas-Shalit [8, 59])**.** *Monomial ideals, and all quasihomogeneous ideals in two variables, have the stable division property.*

As a consequence, we obtained a new unified proof that recovers the result that monomial submodules [6, 15], as well as all quasihomogeneous ideals in two variables [16], are essentially normal. In the paper with Biswas, we make use of the results of Fang and Xia [23, 24]. We use their results also to deduce directly that any ideal in two variables is essentially normal.

Arveson's conjecture actually deals with vector valued spaces of functions. Another contribution I made here was the reduction of the entire (vector valued) problem to the problem of determining whether or not all ideals generated by *scalar valued quadratic forms* satisfy Arveson's conjecture (see the closing section of [59]).

The problem of deciding whether a given ideal has the (approximate) stable division property seems very difficult. I have spent some time trying to use methods from various fields, as well as trying to interest researchers and students, but further progress along these lines seems unlikely (some examples and discussion in our paper might explain why).

Following my paper [59], Kennedy proved that the conjecture holds for submodules that are closed algebraic sums of essentially normal submodules. This, in turn, led some new cases where we showed that the conjecture holds. One of my best results in this field, subsequently obtained jointly with Kennedy, is the proof that the conjecture holds for ideals corresponding to a variety that is a union of subspaces. Such varieties are never smooth except in the most trivial case, and as of now, there are no other methods in the field that can recover our theorem.

Theorem 23 (Kennedy-Shalit [34])**.** *Let $I \lhd \mathbb{C}[z]$ be the radical ideal corresponding to a union of subspaces. Then $M = \overline{I} \subseteq H^2_d$ is essentially normal.*

In another joint work with Kennedy, we introduced operator algebraic methods related to the noncommutative Shilov boundary. Our work has not led to new special cases where the conjecture holds, rather, we proved that some consequences of the conjecture hold true for *all* ideals. Besides the result, of its own interest, this provides strong evidence for the conjecture.

Theorem 24 (Kennedy-Shalit [35])**.** *Let $I \lhd \mathbb{C}[z]$ be a homogeneous ideal. Let S^I denote the shift compressed to $H^2_d \ominus I$. Then the essential norm of every element $f \in \overline{\text{alg}}(S^I)$ is equal to $\sup_{z\in V(I)} |f(z)|$. Moreover, the C^*-envelope of $\overline{\text{alg}}(S^I)/\mathcal{K}$ (as a subalgebra of $C^*(S^I)/\mathcal{K}$) is equal to $C(\partial V(I))$.*

Here

$$V(I) = \{z \in \mathbb{B}_d : p(z) = 0 \text{ for all } z \in I\}$$

and

$$\partial V(I) = \{z \in \partial\mathbb{B}_d : p(z) = 0 \text{ for all } z \in I\}.$$

Identifying the C*-envelope allowed us to prove the following result, which puts the problem in a new perspective.

Theorem 25 (Kennedy-Shalit [35])**.** *Let $I \lhd \mathbb{C}[z]$ be a homogeneous ideal. Let S^I denote the shift compressed to $H^2_d \ominus I$. Then S^I is essentially normal if and only if $\overline{\text{alg}}(S^I)$ is hyperrigid.*

5. Dilation theory in finite dimensions [38, 41, 44, 62]

5.1. **N-dilations on finite dimensional subspaces.** Let $T_1, \ldots, T_k$ be a k-tuple of commuting contractions on a Hilbert space H. A k-tuple $U_1, \ldots, U_k$ of commuting unitaries on a Hilbert space K is said to be a *unitary dilation for* $T_1, \ldots, T_k$, if H is a subspace of K, and if for all $n_1, \ldots, n_k \in \mathbb{N}$ the operator $T_1^{n_1} \cdots T_k^{n_k}$ is the compression of $U_1^{n_1} \cdots U_k^{n_k}$ onto H, meaning that

$$T_1^{n_1} \cdots T_k^{n_k} = P_H U_1^{n_1} \cdots U_k^{n_k}\big|_H.$$

(Here and below we are using the notation P_H for the orthogonal projection of K onto H). The problem of determining if a k-tuple of contractions has a unitary dilation is well studied, and has had a profound impact on operator theory; see [26, 47] for example. The basic results in the theory are that a unitary dilation always exists when $k = 1$ (Sz.-Nagy's unitary dilation theorem) and when $k = 2$ (this is Ando's dilation theorem); also, Parrott gave an example showing that when $k = 3$ a unitary dilation might not exist.

In case there is a dilation, it can be shown that if T_i is not a unitary for some i, then K has to be infinite dimensional. Now, one reason to seek a unitary dilation for a given k-tuple of commuting operators, is to better understand $T_1, \ldots, T_k$ as a "piece" of the k-tuple $U_1, \ldots, U_k$ — given that k-tuples of commuting unitaries are particularly well understood. On the other hand, in the case when $T_1, \ldots, T_k$ act on a finite dimensional space, it is not entirely clear that unitaries acting on an infinite dimensional space are really better understood. One is naturally led to consider a dilation theory that involves only finite dimensional Hilbert spaces.

Definition 26. *Let $T_1, \ldots, T_k$ be commuting contractions on H, and let $N \in \mathbb{N}$. A* unitary N-dilation for $T_1, \ldots, T_k$ *is a k-tuple of commuting unitaries $U_1, \ldots, U_k$ acting on a space $K \supseteq H$ such that*

$$T_1^{n_1} \cdots T_k^{n_k} = P_H U_1^{n_1} \cdots U_k^{n_k} P_H, \tag{6}$$

for all non-negative integers $n_1, \ldots, n_k$ satisfying $n_1 + \ldots + n_k \leq N$.

In 1954 Egerváry showed [20] that every contraction T on a finite dimensional space has a unitary N-dilation acting on a finite dimensional space. In [37] this idea was revisited and some consequences were explored[1]. For a short while I was intrigued by the question: does there exist a finite dimensional version of Ando's theorem? In [41], John McCarthy and I solved this problem.

Theorem 27 (McCarthy-Shalit [41])**.** *Let H be a Hilbert space, $\dim H = n$. Let $T_1, \ldots, T_k$ be commuting contractions on H. The following are equivalent:*

1. *The k-tuple $T_1, \ldots, T_k$ has a unitary dilation.*
2. *For every N, the k-tuple $T_1, \ldots, T_k$ has a unitary N-dilation that acts on a finite dimensional space.*

When the conditions hold, the unitary N-dilation can be taken to act on a Hilbert space of dimension $n^2(n+1)\frac{(N+k)!}{N!k!} + n$.

[1] And this has led me to discover an extremely cute proof of the maximum modulus principle [62]. Not exactly research, but I am proud of that little note.

It follows immediately that every pair of commuting contractions on a finite dimensional Hilbert space has a unitary N-dilation acting on a finite dimensional Hilbert space.

It should be remarked that my motivation for proving the existence of a finite dimensional unitary N-dilation for every commuting pair of contractive matrices, was to use it to obtain a proof of Ando's inequality that does not pass through Ando's dilation theorem, and hence does not use infinite dimensional spaces. Indeed, Ando's inequality for matrices is a statement about matrices, and to my eyes it was remarkable that there was no proof of this inequality that did not go through infinite dimensional spaces. However, the way we ended up proving the existence of N-dilations, was to obtain it as a consequence of Theorem 27 together with Ando's dilation theorem. Thus our proof does not avoid infinite dimensional spaces.

The techniques that McCarthy and I introduced were used later in [9, 10]. In particular, David Cohen, who wrote a master's thesis under my supervision, proved the following result.

Theorem (Cohen [9]). *Let $T = (T_1, \ldots, T_k)$ be a commuting tuple of operators acting on a finite dimensional Hilbert space H, and fix a compact set $X \subseteq \mathbb{C}^k$. The following are equivalent.*

(1) *T has a normal dilation $U = (U_1, \ldots, U_k)$ such that $\sigma(U) \subseteq X$.*
(2) *For every N, T has a normal N-dilation $U = (U_1, \ldots, U_k)$, acting on a finite dimensional Hilbert space, such that $\sigma(U) \subseteq X$.*

Here $\sigma(U)$ denotes the joint spectrum of $U = (U_1, \ldots, U_k)$. If one takes $X = \mathbb{T}^k$, then a normal dilation with spectrum in X is just a commuting unitary tuple, and thus we recover Theorem 27.

5.2. **Spectral sets and distinguished varieties in the symmetrized bidisc.** Ando's dilation theorem (the fact that every pair of commuting contractions on a Hilbert space has a commuting unitary dilation) immediately implies the following inequality:

$$\|f(A, B)\| \leq \|f\|_\infty,$$

for every pair of commuting contractions A, B and every polynomial f, where the supremum norm is computed on the bidisc $\overline{\mathbb{D}}^2$. When A and B are matrices, Agler and McCarthy proved the following sharpening of Ando's inequality.

Theorem (Agler-McCarthy [2]). *Let T_1, T_2 be two commuting contractive matrices, neither of which has eigenvalue of unit modulus. Then there is a one dimensional variety $V \subseteq \mathbb{D}^2$ such that for every polynomial f in two variables*

$$\|f(T_1, T_2)\| \leq \sup_{(z_1, z_2) \in V} |f(z_1, z_2)|.$$

Moreover, Agler and McCarthy prove that V is a so-called *distinguished variety*, i.e., V exists the bidisc through the distinguished boundary $\mathbb{T}^2$.

In [44], Sourav Pal (my former postdoc) and I proved the exact analogue of the above theorem in the setting of the *symmetrized bidisc* Γ, given by

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1|, |z_2| \leq 1\}.$$

A pair of commuting operators (S, P) is said to be a Γ *contraction* if Γ is a spectral set for (S, P).

Theorem 28 (Pal-Shalit [44]). *Let (S, P) be a strict Γ-contraction on a finite dimensional Hilbert space $\mathcal{H}$. Then there is a distinguished variety $W \subset \Gamma$ such that for every matrix valued polynomial f, the following inequality holds*

$$\|f(S, P)\| \leq \max_{(s,p)\in W} \|f(s, p)\|.$$

Since not every Γ-contraction is obtained as the symmetrization of a pair of commuting contractions, the proof is more involved than "pushing forward" the Agler-McCarthy theorem, and contains some nice twists.

We also characterize all distinguished varieties in the symmetrized bidisc, and show that a subvariety of Γ is distinguished if and only if it has a very particular representation as a determinantal variety.

6. Interpolation of completely positive maps and inclusions of matrix convex sets [10]

Suppose one is given two d-tuples of bounded operators $A = (A_1, \ldots, A_d)$ and $B = (B_1, \ldots, B_d)$ on a Hilbert space H. Then the interpolation problem is: *when does there exist a UCP map $\varphi : C^*(A) \to B(H)$ such that $\varphi(A_i) = B_i$ for all i?*

For d-tuple A, we define its *matrix range* to be the free set

$$\mathcal{W}(A) = \cup_n \{\psi(A) : \psi : C^*(A) \to M_n \text{ is UCP}\}.$$

Following Arveson [3], we showed in [10] the following

Theorem 29 (Davidson, Dor-On, Shalit and Solel [10]). *There exists a UCP map $\varphi : C^*(A) \to B(H)$ sending A_i to B_i $(i = 1, \ldots, d)$ if and only $\mathcal{W}(B) \subseteq \mathcal{W}(A)$. There exists a unital completely isometric map sending A_i to B_i $(i = 1, \ldots, d)$ if and only $\mathcal{W}(B) = \mathcal{W}(A)$.*

From this we were able to deduce the criterion for interpolation from [30], as well as other interpolation results (e.g., from [39]).

To what extent does $\mathcal{W}(A)$ determine the spatial structure of the operator tuple A? Under a natural minimality condition, we have the following criterion.

Theorem 30 (Davidson, Dor-On, Shalit and Solel [10]). *Let A and B be* minimal *d-tuples of compact operators. Then $\mathcal{W}(B) = \mathcal{W}(A)$ if and only if A and B are unitarily equivalent.*

We have examples that show that in general, the above theorem fails for non-compact operators. However, we do have the following counterpart that works for the opposite extreme.

Theorem 31 (Davidson, Dor-On, Shalit and Solel [10]). *Let A and B be d-tuples of operators such that*

(1) *$C^*_{env}(A) = C^*(A)$ and $C^*_{env}(B) = C^*(B)$.*
(2) *$C^*(A)$ and $C^*(B)$ contain no nonzero compact operator.*

Then $\mathcal{W}(B) = \mathcal{W}(A)$ if and only if A and B are approximately unitarily equivalent.

Theorem 29 and its consequences led us to investigate inclusion problems for matrix convex sets. A *matrix convex set* is a set of the form $\mathcal{S} = \cup_{n\geq 1}\mathcal{S}_n$, where each $\mathcal{S}_n$ is a set of d-tuples of $n \times n$ matrices, that is invariant under UCP maps from M_n to M_k and under formation of direct sums. As was explained in [30, 31], inclusion problems for matrix convex sets are very well motivated by the theory of robust control.

A natural question, inspired by results from [31], is the following: *if $\mathcal{S}, \mathcal{T}$ are two matrix convex sets, and $\mathcal{S}_1 \subseteq \mathcal{T}_1$, what can we say about the inclusion of $\mathcal{S}$ in $\mathcal{T}$?* To answer this question we studied the minimal and maximal matrix convex sets over a set $K \subseteq \mathbb{R}^d$ (these sets are denoted $\mathcal{W}^{min}(K)$ and $\mathcal{W}^{max}(K)$ below).

Theorem 32 (Davidson, Dor-On, Shalit and Solel [10])**.** *Suppose that $K \subseteq \mathbb{R}^d$, with some nice symmetry properties. Then $\mathcal{W}^{max}(K) \subseteq d\mathcal{W}^{min}(K)$. In particular, if $\mathcal{S}_1 \subseteq \mathcal{T}_1 = K$, then*

$$\mathcal{S} \subseteq d\mathcal{T}.$$

Of course, we have a large supply of examples having the "nice symmetry properties", such as the ball, the cube, and many polytopes. The above theorem is proved by using the characterization of $\mathcal{W}^{min}(K)$:

$$\mathcal{W}^{min}(K) = \{X : \exists N \text{ normal } \sigma(N) \subseteq K \text{ and } X \text{ is a compression of } T\},$$

together with the following dilation result.

Theorem 33 (Davidson, Dor-On, Shalit and Solel [10])**.** *Suppose that $K \subseteq \mathbb{R}^d$, with some nice symmetry properties. For every $A \in B(\mathcal{H})^d_{sa}$ such that the numerical range $\mathcal{W}_1(A) \subseteq K$, there is a d-tuple of* **commuting normal** *operators N on some Hilbert space $\mathcal{K}$ such that*

$$\sigma(N) \subseteq K$$

and such that N is a dilation of $\frac{1}{d}A$. If $\dim \mathcal{H} < \infty$ *then* $\dim \mathcal{K} < \infty$.

We show that the constant d is sharp for a specific matrix convex set $\mathcal{W}^{\max}(\overline{\mathbb{B}_d})$ constructed from the unit ball $\mathbb{B}_d$. This led us to find an essentially unique self-dual matrix convex set $\mathcal{D}$, the self-dual matrix ball, for which corresponding inclusion and dilation results hold with constant $C = \sqrt{d}$.

For a certain class of polytopes, we obtain a considerable sharpening of such inclusion results involving polar duals. An illustrative example is, that a sufficient condition for $\mathcal{T}$ to contain the free matrix cube $\mathfrak{C}^{(d)} = \cup_n \{(T_1, \dots, T_d) \in M_n^d : \|T_i\| \leq 1\}$, is that $\{x \in \mathbb{R}^d : \sum |x_j| \leq 1\} \subseteq \frac{1}{d}\mathcal{T}_1$, i.e., that $\frac{1}{d}\mathcal{T}_1$ contains the polar dual of the cube $[-1,1]^d = \mathfrak{C}_1^{(d)}$.

In a recent work [46], we consider different sets K and search for the best constant $\theta(K)$ such that $\mathcal{W}^{\max}(K) \subseteq \theta(K)\mathcal{W}^{\min}(K)$. Here are two of our main results.

Theorem 34 (Passer-Shalit-Solel [46])**.** $\theta(K) = 1$ *if and only if K is a simplex.*

Theorem 35 (Passer-Shalit-Solel [46])**.** *Let $\overline{\mathbb{B}}_{p,d}$ be the unit ball in $\mathbb{R}^d$ with the ℓ^p norm. Then $\theta(\overline{\mathbb{B}}_{p,d}) = d^{1-|1/2-1/p|}$.*

Both results improve on known results. For $p \neq 2$, in particular for $p = 1, \infty$, the sharp constant is new. Theorem 34 improves a result of Fritz, Netzer and Thom [25], who proved this for the case where K is a polytope.

7. Functional equations, dynamical systems and applications to partial differential equations [54, 57, 61]

This is the topic on which I wrote my master's thesis [53] (available on the arxiv). I kept up my interest in it for a few years, and here I will present one pretty result.

Consider a function $f : [-1,1] \to \mathbb{R}$ that satisfies the following functional equation:

$$f(t) = f(\delta_1(t)) + f(\delta_2(t)) \ , \ t \in [-1,1]. \tag{7}$$

where δ_1, δ_2 are self maps of $[-1,1]$ forming a *regular $\mathcal{P}$-configuration*; meaning that δ_1, δ_2 are continuously differentiable, with non vanishing derivatives, $\delta_1(-1) = -1$, $\delta_1(1) = 0 = \delta_2(-1)$ and $\delta_2(1) = 1$, and $\delta_1(t) + \delta_2(t) = t$. As a representative case, consider the maps

$$\delta_1(t) = \frac{t+1}{2} \quad , \quad \delta_2(t) = \frac{t-1}{2}.$$

Such functional equations arose in Paneah's study of hyperbolic PDEs [45], and have also been considered from the pure functional equations viewpoint.

If $f \in C^1$ (i.e., if f is continuously differentiable) is a solution to (7), then it is a nice exercise to show that f must have the form $f(x) = cx$. What if f is only assumed to be continuous? Are there any other continuous solutions?

Theorem 36 (Shalit [57])**.** *Let δ_1, δ_2 form a regular $\mathcal{P}$-configuration. Then the functional equation* (7) *has continuous solutions that are not C^1.*

The theorem is important, because it shows (roughly) that the differentiability conditions required in Paneah's theory (as well as in my subsequent developments) are sharp. Interestingly, the proof goes by first showing that all regular $\mathcal{P}$-configurations are conjugate, as dynamical systems. In contrast, later in that same paper I show that there are infinitely many $\mathcal{P}$-configurations whose derivatives have exactly one point of vanishing, which are mutually non-isomorphic (as *guided dynamical systems*).

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