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Role	Name	Academic Rank	Department	Institute
PI.1	Shalit Orr Moshe	Assistant Professor	Mathematics	Technion

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Application No.: 195/16
PI name: Orr Moshe Shalit

Scientific abstract: **Operator-algebraic geometry in noncommutative analysis**

The goal of this proposal is to study the interplay between operator theory, function theory, and the spaces on which functions act — in the noncommutative setting. The key objects of study are noncommutative (nc) functions and varieties in the sense in which they appear in recent works of several groups, e.g.: Kalyuzhnyi-Verbovetskii and Vinnikov, Agler and M$^{\mathrm{c}}$Carthy, and Helton, Klep and McCullough.

A motivating observation is that algebras of bounded nc analytic functions can be thought of not only as algebras of "functions" which one my apply to suitable operators, but also as operator algebras in their own right. In particular, I plan to study the operator algebraic structure of the algebras of bounded analytic functions on nc varieties, and investigate the connection to the geometric structure of the nc varieties. In addition, I aim at revealing the spatial features of these algebras as multiplier algebras of reproducing kernel Hilbert spaces of nc functions, and the interplay of all of the above with the function theoretic properties of elements in the algebras. Specifically, the goals I propose are:

(1) To identify algebras of bounded nc analytic functions on nc domains and varieties as operator algebras and study their structure. Conversely, to represent operator algebras as algebras of nc functions and classify the algebras up to isomorphism by the geometric properties of the nc varieties on which they live.
(2) To study nc reproducing kernel Hilbert spaces. In particular, to prove an Agler-M$^{\mathrm{c}}$Carthy type embedding theorem that characterizes all nc complete Nevanlinna-Pick Hilbert function spaces as subspaces of the Fock space on the matrix unit ball, and use this point of view to classify their multiplier algebras.
(3) To investigate approximation and extension problems for nc analytic functions on nc varieties. In the same vein, to consider noncommutative versions of the Nullstellensatz.
(4) To use the results in the noncommutative setting to shed light and clarify problems and results in the classical (commutative) case.
(5) To apply methods of matrix convexity and noncommutative analysis to the interpolation problem for unital completely positive (UCP) maps, and to apply the results to study the existence and uniqueness of UCP maps with certain properties between operator spaces. Lastly, to apply these results to noncommutative Choquet theory in operator algebras.

This proposal is a natural continuation of my work under an ISF grant no. 474/12 (2012–2016), and builds upon my success (together with collaborators) to solve analogous problems in the commutative case. I expect to obtain beautiful and interesting results, that will tie together many different fields such as operator theory, matrix theory, complex analysis, operator algebras, algebraic geometry and abstract algebra.

Detailed description of the research program

1. Scientific Background

1.1. **Overview.** The goal of this proposal is to study the interplay between operator theory, function theory, and the spaces on which functions act — in the noncommutative setting. Personally, this interplay fascinated me since I learned Gelfand's theory, and how, when the Gelfand transform is turned on its head, it gives rise to von Neumann's spectral theory and functional calculus. Many (and better) others have been fascinated by this connection, and, since the times of von Neumann and Gelfand, functional calculi have been developed beyond the limited setting of commuting normal operators. A small sample of classical developments:

(1) The (Riesz-Dunford) holomorphic functional calculus for a single operator.
(2) The (Sz.-Nagy–Foias) H^∞ functional calculus (see [28]).
(3) The (Shilov-Arens-Calderón) holomorphic functional calculus for several elements in a commutative Banach algebra.
(4) The Taylor functional calculus for several commuting operators on a Banach space (see [72, 73]).

(See [30] for a useful overview, or [62, 76] for a broader view).

In the early 1970s, immediately after he finished constructing the *Taylor spectrum* and the *Taylor functional calculus*, J.L. Taylor started developing a framework for a functional calculus that works for *noncommuting* operators [74, 75]. Unlike the cases of the above mentioned functional calculi, in the case of noncommuting tuples of operators, it is not clear what are the "functions" to be applied to operators. Clearly one can apply polynomials in noncommuting variables, but there was no available class of analytic functions for which a functional calculus could be developed; the noncommutative functional *calculus* was developed by Taylor together with the noncommutative *functions* themselves.

In passing from the commutative to the noncommutative theory, Taylor's point of view was that the class of algebras $\mathcal{O}(U)$ of holomorphic functions on a domain $U \subseteq \mathbb{C}^d$, should be considered as a "relatively small, well understood and nicely behaved class of topological algebras with distinguished tuples $(z_1, \ldots, z_d)$ of elements" that serve as models for the behaviour of tuples of elements in a commutative Banach algebra. This leads to the question: what finitely generated topological algebras can serve as models for the behaviour of tuples of element in a (noncommutative) Banach algebra? Naturally, Taylor's starting point was the free algebra generated by d (noncommuting elements) — that is, polynomials in d noncommuting variables. The "localizations" — certain topological closures of this free algebra (in a sense Taylor makes precise) — are the algebras of noncommutative functions that one would be able to apply to certain tuples [74, 75].

Taylor's methods and intuition were based on ideas from homological algebra, and since then more concrete versions of noncommutative function theory have been developed. In the last decade (due to public demand!) the subject developed rapidly. Influential references on these developments are the monograph [52] by Kalyuzhnyi-Verbovetskii and Vinnikov, and the papers[77, 78] by Voiculescu; one should also be aware of various contributions by many authors, such as Agler-M$^{\text{c}}$Carthy (e.g. [3, 4, 5]), Alpay and Kalyuzhnyi-Verbovetzkii (e.g. [8]), Ball-Groenewald-Malakorn (e.g. [24]), Helton-Klep-McCullough (e.g. [42]), Muhly-Solel (e.g. [58, 59, 60, 61]), Popescu (e.g. [67, 68]), and others (e.g. [63]).

My goal is to study a class of algebras that may be called *noncommutative function algebras*. In the joint spirits of Gelfand and Taylor, I will view noncommutative functions both as functions and as operators in certain operator algebras, simultaneously. In this research proposal I will rely on a natural and concrete version of noncommutative function theory that has been quite successful in recent years, and which fits well with my past and my current research objectives.

In short, a *noncommutative (nc) analytic function in d variables* is a function from d-tuples of $n \times n$ matrices in some "nc domain" to $n \times n$ matrices, that respects direct sums and similarity (see Section 1.2). The connection with Taylor's point of view is that such functions are locally approximable by free polynomials [3], thus the algebra of such functions is a kind of "localization" that Taylor considered. This fits within the framework that Kalyuzhnyi-Verbovetskii and Vinnikov laid, but I will limit attention to the case where the base scalar field is the complex numbers (i.e., the arguments of the functions are plain matrices, rather than matrices over arbitrary modules or operator algebras). This is a good point to stress that I choose to concentrate not on the most general version of nc function theory, in order to make progress on hard questions in analysis and operator algebras which are unlikely to have clean or interesting solutions in full generality (cf. complex algebraic geometry *vs.* Algebraic Geometry).

One of the observations motivating this proposal, is that algebras of *bounded* nc analytic functions can be thought of not only as algebras of "functions" which one may apply to suitable operators, but also as operator algebras in their own right. Based on this observation, I propose to study algebras of (bounded) nc functions from an operator algebraic perspective, with an emphasis on algebras of functions on nc *varieties*. In particular, I plan to study the operator algebraic structure of the algebras of bounded analytic functions on nc varieties, and investigate the connection to the geometric structure of the nc varieties. In addition, I aim at revealing the spatial features of these algebras as multiplier algebras of reproducing kernel Hilbert spaces of nc functions, and the interplay of all of the above with the function theoretic properties of elements in the algebras.

Closely related objects that we also wish to consider are nc *semi-algebraic sets*, that is, nc sets determined not by polynomial equations, but rather by matrix valued polynomial inequalities. Research of Helton, Klep, McCullough and others (see [42, 43, 44, 45, 46, 47]), as well as preliminary work of the author with Davidson, Dor-On and Solel [31], show that this is closely connected to problems of existence and uniqueness of completely positive maps that map between two given sets of operators.

This proposal is a natural continuation of my work under an ISF grant no. 474/12 (2012–2016). A successful part of that project was the research on the "isomorphism problem", and is a specialization to the commutative case of several of the main problems proposed below.

1.2. **Noncommutative function theory.** I will now introduce the terminology and notation for the proposal. (When a result is given without reference or explanation, the proof can be found in [52]).

Let $M_n = M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices over $\mathbb{C}$, and let M_n^d be the set of all d-tuples of such matrices. The universal underlying set on which all noncommutative functions in d variables feed (the "noncommutative universe") is the disjoint union

$$\mathbb{M}^{[d]} = \cup_{n=1}^{\infty} M_n^d.$$

A subset $\Omega \subseteq \mathbb{M}^{[d]}$ is called a *free set*. A free set Ω is said to be an *nc set* if it is closed under direct sums and conjugation with unitaries. If Ω is an nc set, we denote $\Omega_n = \Omega \cap M_n^d$. A function f from an nc set $\Omega \subseteq \mathbb{M}^{[d]}$ to $\mathbb{M}^{[1]}$ is said to be an *nc function* if

(1) f is graded: $X \in \Omega_n \Rightarrow f(X) \in M_n$,
(2) f respects direct sums: $f(X \oplus Y) = f(X) \oplus f(Y)$,
(3) f respects similarities: if $X \in \Omega_n$ and $S \in M_n$ is invertible, and if $S^{-1}XS \in \Omega_n$, then $f(S^{-1}XS) = S^{-1}f(X)S$.

A *free polynomial* is an element in $\mathbb{C}\langle z_1, \ldots, z_d\rangle$ (the free algebra in d variables). Free polynomials serve as the most important example of nc functions.

There are at least four interesting different ways in which $\mathbb{M}^{[d]}$ can be topologized (see [6]); I will mention two, and concentrate only on the free topology. A free set Ω is said to be open in the *du (= disjoint union)* topology if Ω_n is open in M_n^d for all n. The *free* topology is the topology generated by *basic free open sets*, which are sets of the form

$$G_\delta := \{X \in \mathbb{M}^{[d]} : \|\delta(X)\| < 1\},$$

where δ is a matrix of free polynomials. For example, the *(d-dimensional) open matrix unit ball* $\mathfrak{B}_d$ is defined to be G_δ where $\delta(z) = [z_1\ z_2\ \ldots\ z_d]$. Thus $\mathfrak{B}_d = \{X \in \mathbb{M}^{[d]} : \|\sum X_j X_j^*\| < 1\}$.

An nc function defined on a free open set Ω is said to *free analytic* (or *free holomorphic*) if it is locally bounded (here, the dependence on the free topology enters in the use of the word "locally"). When the topology is unspecified we may say that the function is *nc analytic*. It turns out that an nc analytic function is really an analytic function when considered as a function $f : \Omega_n \to M_n$, for all n, and moreover it has a "Taylor series" at every point. A function $f : \Omega_n \to M_n$ is free holomorphic if and only if it is locally approximable by polynomials [3].

A *noncommutative (nc) algebraic variety* is a free set of the form $V_\Omega(S) = \{X \in \Omega : \forall p \in S\ .\ p(X) = 0\}$, where $S \subseteq \mathbb{C}\langle z_1, \ldots, z_d\rangle$. Likewise, let us define an *nc analytic variety in* Ω to be the joint zero set of a set of nc analytic functions on Ω (there are potentially more general definitions that may be worth considering, but this will be the working definition for the start). Note that nc algebraic and analytic varieties are nc sets.

Finally, if Ω is an open nc set, $V \subset \Omega$ is an nc variety, and $f : V \to \mathbb{M}^{[d]}$ is a function, we say that f is *nc analytic* if for every $v \in V$ there is some neighbourhood $U \ni v$ and an nc analytic function $F : U \to \mathbb{M}^{[1]}$ such that $F\big|_{U\cap V} = f\big|_{U\cap V}$. We define $H^\infty(\Omega)$ to be the algebra of bounded holomorphic functions on Ω, and $A(\Omega)$ to be the algebra of bounded analytic functions that extend continuously to $\partial\Omega$ (and likewise for varieties).

1.3. **Other versions of noncommutative function theory.** There are other version of noncommutative function theory that I wish to keep in mind and compare with the above. Popescu [67] studied functions with a power series representation defined on the unit ball of operators on a Hilbert space. Agler and McCarthy [4] study *intertwining preserving* functions on *operator domains*, and show that, if such a function also satisfies a certain continuity condition, then it defines a free analytic function; Agler and McCarthy also show that the converse holds in some cases.

1.4. **The noncommutative disc algebra and the noncommutative analytic Toeplitz algebra.** Let E be a d-dimensional Hilbert space. The *full Fock space* is the space

$$\mathcal{F}_d = \mathcal{F}(E) = \mathbb{C} \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \dots .$$

Fix a basis $\{e_1, \dots, e_d\}$ of E. On $\mathcal{F}_d$ define $L = (L_1, \dots, L_d)$ by

$$L_i x_1 \otimes \cdots \otimes x_n = e_i \otimes x_1 \otimes \cdots \otimes x_n.$$

L is called the *noncommutative d-shift.* The tuple L is easily seen to be a *row isometry*, that is, $L_1, \dots, L_d$ are isometries with pairwise orthogonal ranges. The tuple L plays a central role in noncommutative multivariable operator theory, see, e.g., [16, 34, 35, 36, 64, 65, 66].

The *noncommutative disc algebra* $\mathfrak{A}_d$ and the *noncommutative analytic Toeplitz algebra* $\mathcal{L}_d$ are defined to be the norm closed algebra and the weak-operator closed algebra, respectively, generated by L. These algebras were introduced by G. Popescu in [65], where it was shown that $\mathcal{L}_d$ is the noncommutative multiplier algebra of the full Fock space.

The noncommutative disc algebra is the universal operator algebra generated by a row contraction in the following sense: for every row contraction $T = (T_1, \dots, T_d) \in B(H)^d$ there exists a unital, completely contractive homomorphism $\varphi : \mathfrak{A}_d \to B(H)$ such that $\varphi(L_i) = T_i$ for all i.

Combining [4, Theorem 5.7] and [67, Theorem 3.1] one finds that $\mathcal{L}_d$ can be identified completely isometrically isomorphically with $H^\infty(\mathfrak{B}_d)$ — the algebra of bounded free analytic functions on $\mathfrak{B}_d$. On the other hand, it is not clear whether or not $\mathfrak{A}_d$ is natrually isomorphic to $A(\mathfrak{B}_d)$.

1.5. **Universal algebras for relations and their WOT closures.** In [71], B. Solel and I considered universal tuples for homogeneous ideals in noncommuting variables (rediscovering a few results of Popescu [66]), and this led in [37, 40] to the classification up to isometric isomorphism of universal operator algebras generated by a commuting row contraction satisfying the relations in a homogeneous ideal, as well as their WOT closures. The results can be briefly summarized as follows.

For a homogeneous ideal $I \triangleleft \mathbb{C}\langle z_1, \dots, z_d \rangle$, denote by $\mathcal{A}_I$ the universal operator algebra generated by a row contraction S such that $p(S) = 0$ for all $p \in I$. Abstractly, $\mathcal{A}_I$ can be considered as the quotient of the $\mathfrak{A}_d$ by the closed ideal generated by elements of the form $p(L)$, where $p \in I$ (see [50, Proposition 3.1]). Concretely, $\mathcal{A}_I$ is the norm closed operator algebra acting on $\mathcal{F}_d \ominus I$ (where I is identified as a subspace of the Fock space $\mathcal{F}_d$ by identifying noncommutative polynomials as elements of $\mathcal{F}_d$), generated by the "constrained shift" $S^I := P_{I^\perp} L P_{I^\perp}$ — the the compression of the shift L to $\mathcal{F}_d \ominus I$ (see [66, 71]). When I is the ideal of commutators then one gets Arveson's algebra $\mathcal{A}$ (or $\mathcal{A}_d$) from [19]. The first classification result is that $\mathcal{A}_I$ and $\mathcal{A}_J$ are isometrically isomorphic if and only if

I and J are related by a unitary change of variables (and this happens if and only if the *subproduct systems* associated to I and J are isomorphic). If we denote by $\mathcal{M}_I$ the WOT closure of $\mathcal{A}_I$ (in the concrete picture), then $\mathcal{M}_I$ and $\mathcal{M}_J$ are isometrically isomorphic under exactly the same terms.

The same results hold true when one specializes to commutative variables, but in the commutative case one can say more. For an ideal $I \triangleleft \mathbb{C}[z_1, \ldots, z_d]$, let $V_{\mathbb{B}_d}(I) = \{z \in \mathbb{B}^d : \forall p \in I . p(z) = 0\}$. If I and J are *radical* homogeneous ideals in $\mathbb{C}[z_1, \ldots, z_d]$, then $\mathcal{A}_I$ is isometrically isomorphic to $\mathcal{A}_J$ if and only if there is unitary mapping $V_{\mathbb{B}_d}(I)$ onto $V_{\mathbb{B}_d}(J)$; moreover, $\mathcal{A}_I$ and $\mathcal{A}_J$ are isomorphic as algebras if and only if the varieties $V_{\mathbb{B}_d}(I)$ and $V_{\mathbb{B}_d}(J)$ are biholomorphically equivalent. The same is true for the algebras of the type $\mathcal{M}_I$.

Related works in the noncommutative setting are [12, 13, 14, 15, 50, 57, 68, 69].

1.6. **Interpolation problems for completely positive maps and free spectrahedra.** The "noncommutative universe" $\mathbb{M}^{[d]}$ is also very useful for studying problems that can be considered as "noncommutative real algebraic geometry" such as the representation of positive nc functions as sums of squares [42, 43, 44, 45, 46, 47]. Such problems were shown by Helton, Klep and McCullough to be closely related to interpolation problems of unital completely positive (UCP) maps [44]. Interpolation problems for UCP maps, in turn, have been of great interest, because of their relationship to quantum information theory and theoretical physics, as well as with operator algebras [9, 10, 29, 41, 48, 49, 55].

The problem, which I have recently taken up with collaborators [31], can be stated as follows. Suppose one is given two d-tuples of bounded operators $A = (A_1, \ldots, A_d)$ and $B = (B_1, \ldots, B_d)$. Then the interpolation problem is: *when does there exist a UCP map* $\varphi : C^*(A) \to C^*(B)$ *such that* $\varphi(A_i) = B_i$ *for all* i*?* (Below we will usually abbreviate this as "there exists a UCP map $\varphi : A \mapsto B$").

One can characterize the existence of a UCP map $\varphi : A \mapsto B$ using *nc semi-algebraic sets.* Indeed, define

$$\mathcal{D}_A = \{X \in \mathbb{M}^{[d]} : \operatorname{Re} \sum A_j \otimes X_j \leq I\}.$$

Note that $\mathcal{D}$ is simply the set where a *linear matrix valued* nc function (also called a *linear pencil*) — namely, the function $L(X) = I - \sum A_j \otimes X_j$ — obtains values with positive real part; hence the terminology. By a result that essentially appears in [44], assuming that $\mathcal{D}_A$ is bounded, such a map exists if and only if $\mathcal{D}_A \subseteq \mathcal{D}_B$.

For d-tuple A, we define its *matrix range* to be the free set

$$\mathcal{W}(A) = \cup_n \{\psi(A) : \psi : C^*(A) \to M_n \text{ is UCP}\}.$$

Following Arveson [18], we showed in [31] that there exists a UCP map $\varphi : A \to B$ if and only $\mathcal{W}(B) \subseteq \mathcal{W}(A)$. We also observed that $\mathcal{W}(A)$ and $\mathcal{D}_A$ are related by the Effros-Winkler *matricial polar duality* [39] via $\mathcal{W}(A)^\circ = \mathcal{D}_A$, and from this we were able to deduce the above mentioned criterion for interpolation from [44]. Other interpolation results (e.g., from [55]) also follow from our criterion.

These results led us to investigate inclusion problems for matrix convex sets. A central question, inspired by results from [46], is the following: *if* $\mathcal{S}, \mathcal{T} \subseteq \mathbb{M}^{[d]}$ *are two matrix convex sets, and* $\mathcal{S}_1 \subseteq \mathcal{T}_1$*, what can we say about the inclusion of* $\mathcal{S}$ *in* $\mathcal{T}$*?* More generally, one can ask what are the minimal and maximal matrix convex sets with prescribed properties. Following preliminary results we obtained, it seems worthy to continue to investigate these questions and others regarding matrix convex nc subsets of $\mathbb{M}^{[d]}$ and their applications to the UCP interpolation problem.

2. Research objectives and expected significance

2.1. **Objectives.** The main objectives of this proposal fall under the heading: *to investigate, via nc function theory, the connection between the structure of operator algebras/systems, on the one hand, and the geometry of nc sets and nc varieties, on the other hand.* More specifically, my goals are:

(1) To identify algebras of bounded nc analytic functions on nc domains and varieties as operator algebras and study their structure. Conversely, to represent operator algebras (in particular the algebras of the form $\mathcal{A}_I$ and $\mathcal{M}_I$) as algebras of nc functions and classify the algebras up to isomorphism by the geometric properties of the nc varieties on which they live.
(2) To study nc reproducing kernel Hilbert spaces. In particular, to prove an Agler-McCarthy type embedding theorem that characterizes all nc complete Nevanlinna-Pick Hilbert function spaces as subspaces of the Fock space on the matrix unit ball, and use this point of view to classify their multiplier algebras in the spirit of [38, 70].
(3) To investigate approximation and extension problems for nc analytic functions on nc varieties. In a similar vein, in study noncommutative versions of the Nullstellensatz.
(4) To use the results in the noncommutative setting to shed light and clarify problems and results in the classical (commutative) case.
(5) To apply methods of matrix convexity and noncommutative analysis to the UCP interpolation problem, and to apply these results to study the existence and uniqueness of UCP maps with certain properties between operator spaces. Lastly, to apply these results to noncommutative Choquet theory, and to problems in operator algebras.

2.2. **Significance.** Noncommutative function theory is an active and growing field of research, and I believe that my work will have impact on this emerging field. As explained above, studying the algebraic structure of the algebras of bounded free analytic functions will also shed light on previously investigated isomorphism problems for universal operator algebras.

Among the problems I propose to study, the UCP interpolation problem may have the broadest impact beyond multivariable operator theory, touching upon semidefinite programming, quantum information theory, as well as operator algebras.

3. Detailed description of the proposed research

3.1. **NC analytic functions, analytic functions on operator sets, and universal operator algebras.** In [67], Popescu considered analytic functions on the open unit ball in $B(H)^d$ which are given by power series around the origin. Although it is clear that every such function gives rise to a free analytic function in $\mathfrak{B}_d$, the converse is not transparent. It does in fact follow from [4, Theorem 5.7], [67, Theorem 3.1] and [52, Chapter 7] that every bounded free analytic function on $\mathfrak{B}_d$ gives rise to a bounded analytic function on the unit ball of $B(H)^d$.

One can consider other kinds of "operator domains", and define analytic functions on them, with definitions similar to the definition of nc analyticity on subsets of $\mathbb{M}^{[d]}$. I am interested in studying whether, as in the ball, analytic functions on operator domains arise from nc functions on matrix versions of these domains. Agler and McCarthy have some positive results in this direction [4].

Further, it follows from the papers of Agler-McCarthy and Popescu that $\mathcal{L}_d$ can be identified completely isometrically isomorphically with $H^\infty(\mathfrak{B}_d)$. I plan to determine whether a similar result holds for quotients of $\mathcal{L}_d$ by WOT closed homogeneous ideals. In particular, I conjecture that for every homogeneous ideal $I \triangleleft \mathbb{C}\langle z_1, \ldots, z_d\rangle$, the algebra $\mathcal{M}_I$ is completely isometrically isomorphic with $H^\infty(V_{\mathfrak{B}_d}(I))$. If successful, I will also study the same question for non-homogeneous ideals, or non-homogeneous nc analytic varieties.

The same conjecture can be made for the norm closed algebras $\mathcal{A}_I$, however, this will be a more difficult task. The difficulty here is that first we must determine the relationship between $\mathfrak{A}_d$ and $A(\mathfrak{B}_d)$. If it turns out that $\mathfrak{A}_d \cong A(\mathfrak{B}_d)$, then it would be natural to conjecture that $\mathcal{A}_I \cong A(V_{\mathfrak{B}_d}(I))$ for every homogeneous $I \triangleleft \mathbb{C}\langle z_1, \ldots, z_d\rangle$. However, if $\mathfrak{A}_d$ is not $A(\mathfrak{B}_d)$, then the question arises whether one can represent $\mathfrak{A}_d$ and its quotients as natural algebras of free analytic functions; and if so, what are these algebras?

3.2. **The isomorphism problem for universal operator algebras.** The next problem I propose to attack, is that of obtaining geometric invariants for the operator algebras $\mathcal{A}_I$ and $\mathcal{M}_I$. Let us consider first the commutative case. The results mentioned in Section 1.5 give a very satisfying description, in the commutative radical case, of the algebraic and operator-algebraic structure of the algebra $\mathcal{A}_I$ (or $\mathcal{M}_I$) in terms of the geometry of the variety $V_{\mathbb{B}_d}(I)$ (these results were extended in [32, 38, 54, 56] to operator algebras associated to analytic varieties that are not homogeneous; see the survey [70]). Although it is perhaps forgivable (in perspective of Hilbert's Nullstellensatz) that a geometric picture is available only in the case of radical ideals, one would like to be able to handle non-radical ideals as well. In light of Amitsur's Nullstellensatz [11], nc algebraic varieties seem to be the right generalization of variety to serve as a complete invariant for universal operator algebras for polynomial relations (at least when the relations are homogeneous).

Let us now move back to the noncommutative setting. A good place to start is to examine the nc variety $V_{\mathfrak{B}_d}(I)$. Following the above conjecture that $\mathcal{M}_I = H^\infty(V_{\mathfrak{B}_d}(I))$, and based on results in the commutative case [38, 70], I make the additional conjecture that (for homogeneous ideals I and J), $\mathcal{M}_I \cong \mathcal{M}_J$ if and only if there is an nc biholomorphic map of $V_{\mathfrak{B}_d}(I)$ onto $V_{\mathfrak{B}_d}(J)$. Regardless of whether $\mathcal{A}_I$ turns out to be equal to $A(V_{\mathfrak{B}_d}(I))$ or not, I will seek a geometric classification result, perhaps the nc variety $V_{\mathfrak{B}_d}(I)$ will play a role here too.

3.3. **NC reproducing kernel Hilbert spaces and the isomorphism problem for complete Nevanlinna-Pick nc multiplier algebras.** Above I proposed to investigate the relationship between universal operator algebras generated by tuples satisfying relations in an ideal, and algebras of bounded nc analytic functions on noncommutative varieties. A possible approach to get an operator theoretic handle on algebras of bounded nc analytic functions is via the theory of nc reproducing kernel spaces. Various noncommutative versions have been developed ([8, 27, 60]), and it appears that a free function theoretic version developed by Ball, Vinnikov, and others is the most appropriate [22, 23, 51].

Due to limitations of space, let us just consider an example: the *noncommutative Szegö kernel* [23] (also called the *noncommutative Drury-Arveson kernel*):

$$k_{Sz}(z, w^*) = \sum_{\alpha \in \mathbb{F}_+^d} z^\alpha (w^*)^\alpha.$$

The Fock space $\mathcal{F}_d$ can be considered as Hilbert space of formal power series in d noncommutative indeterminates. Moreover, $\mathcal{F}_d$ is actually a space consisting of nc analytic functions on $\mathfrak{B}_d$ in an obvious way. The kernel k_{Sz} is a noncommutative reproducing kernel for the Fock space $\mathcal{F}_d$, in the sense that

(1) For every fixed $W \in \mathfrak{B}_d$, $k_{Sz}(z, W)$ is a (matrix valued) nc analytic function on $\mathfrak{B}_d$, and vice versa;
(2) for every $h \in \mathcal{F}_d$, $\langle h, k_{Sz}(z, \overline{W})\rangle = h(W)$, when interpreted correctly.

Ball, Marx and Vinnikov [22] showed that this kernel is a complete Pick nc kernel. Further, they conjecture that there is an analogue of the Agler-McCarthy type embedding theorem [1], which characterizes all nc complete Nevanlinna-Pick Hilbert function spaces as subspaces of the Fock space on $\mathfrak{B}_d$, in the sense that for every complete Pick nc kernel K, there exists an nc variety $V \subseteq \mathfrak{B}_d$ for some d, such that K is essentially equal to $k\big|_{V\times V}$. I will consider this conjecture, and, if it is true, I will also study the isomorphism problem for complete Pick nc kernels via nc varieties. The goal will be to obtain noncommutative analogues of results in [32, 38, 54, 70]. For example, if k and k' are such kernels, represented on nc varieties V and V', then I conjecture further that the associated multiplier algebras $\mathrm{Mult}(\mathcal{H}(k))$ and $\mathrm{Mult}(\mathcal{H}(k'))$ are completely isometrically isomorphic if and only if there exists an nc automorphism Θ of $\mathfrak{B}_d$ such that $V' = \Theta(V)$.

3.4. **Infinitely many variables.** In the previous subsection I noted the conjecture of Ball, Marx and Vinnikov that there exists a universal complete Pick nc kernel. However, reflecting on this issue one realizes that one will have to consider the matrix unit ball in infinitely many variables $\mathfrak{B}_\infty$. Thus, I also plan to think about nc function theory in $d = \infty$ variables. This still fits in the framework of Kalyuzhnyi-Verbovetskii and Vinnikov [52], but less is known in this case, and there are sure to be interesting subtleties to explore. For example, inspired the results in [56] it will be interesting to study the properties of the Bohr map (related of Dirichlet series) in the noncommutative setting.

3.5. **Free Nullstellensatz.** Let $J \triangleleft \mathbb{C}\langle z_1, \ldots, z_d\rangle$ be an ideal, and denote by $V_{rc}(J)$ the "noncommutative variety" consisting of all row contractions $T \in B(H)^d$ such that $p(T) = 0$ for all $p \in J$. Further, denote by $I(V_{rc}(J))$ the ideal that consists of all polynomials that vanish on $V_{rc}(J)$. In [71], Solel and I obtained a noncommutative Nullstellensatz, which says that if J is a homogeneous ideal then $I(V_{rc}(J)) = J$. Together with Eli Shamovich I started thinking about an analogous question when $V_{rc}(J)$ is replaced by the nc variety $V_{\mathfrak{B}_d}(J)$, and we proved a noncommutative homogeneous Nullstellensatz in this setting as well. Note that the assumption that J is homogeneous is necessary in both formulations: consider as a counter example the ideal generated by $1 - z_1 z_2 - z_2 z_1$.

In fact, this is a classic type of question treated by Amitsur [11], however he does not specify to homogeneous ideals, hence obtains a result that is deeper, but not as clean. A more analytic variant of this problem is to consider ideals inside an operator algebra instead of in $\mathbb{C}\langle z_1, \ldots, z_d\rangle$. For example, I propose to examine whether or not $I_{\mathfrak{A}_d}(V_{\mathfrak{B}_d}(J)) = \sqrt{J}$, where J is a homogeneous ideal in $\mathfrak{A}_d$, $I_{\mathfrak{A}_d}(V_{\mathfrak{B}_d}(J))$ is the ideal of functions in $\mathfrak{A}_d$ that vanish on $V_{\mathfrak{B}_d}(J)$, and the square root denotes an appropriately defined radical. Such a result was proved in the commutative setting in [37].

We believe that, perhaps using insights from [11], we may also be able to say something about nonhomogeneous ideals. Moreover, if we are able to understand the Nullstellensatz for nc nonhomogeneous ideals, this might also shed light on what happens for nonhomogeneous ideals in the commutative case.

3.6. **Extension and approximation problems.** Given an nc analytic function f in an nc open set, which vanishes on an nc variety V, is it true that f can be approximated (in various senses) by the ideal of all polynomials vanishing on V? This question has an affirmative answer in the commutative homogeneous case [37]. I do not expect to give an affirmative answer in full generality, but I will study both the homogeneous and the nonhomogeneous cases. This may shed light on the nonhomogeneous problem in the commutative setting as well.

A very closely related problem is whether an nc analytic function on an nc subvariety of some nc open set can be extended to an nc analytic function on the open set. Agler and McCarthy found that the answer is "yes" in the case of an *algebraic* variety in a basic free open set [7]. I will consider, as a first step beyond this, the case of extending an nc analytic function from an analytic nc variety in $\mathfrak{B}_d$ to all of $\mathfrak{B}_d$.

3.7. **Other domains.** One of the most interesting aspects of classical function theory in several variables (as opposed to one variable) is that the domain on which one does function theory is not the background — it plays a leading role. The nc varieties in the above problems were all inside the matrix ball $\mathfrak{B}_d$. I will strive to obtain results at least in the generality of free basic open sets of the form G_δ (these include $\mathfrak{B}_d$ and also the matrix polydisc, but of course much more).

3.8. **Interpolation problems for UCP maps and hyperrigidity.** In the previous section I mentioned that nc domains other than the matrix ball will be of interest. A large class of nc domains arise as nc semi-algebraic sets, and in particular as free spectrahedra. This brings us to the last ring of problems in this proposal, which is somewhat different than all of the above, being more like real nc function theory than complex nc function theory.

Recall the interpolation problem for unital completely positive (UCP) maps from Section 1.6. In the literature, most attention was given to the case when A and B are tuples of matrices, and satisfactory solutions exist (see, e.g., [9, 10, 44]). In a recent collaboration [31] we took up the interpolation problem in the generality of bounded operators, with a slightly different (perhaps more conceptual) point of view. For example, we have some new results on when a positive linear map can be "scaled" to become completely positive. Our results are motivated by [46], but in that paper the emphasis was on obtaining scaling constants which depend on the ranks of the data, whereas we concentrate on obtaining rank-independent results.

The problem of scaling a positive map to become completely positive turns out to be closely related to the following problem: *given a convex set K in $\mathbb{C}^d$, what are the minimal and maximal matrix convex sets $\mathcal{S}^{min}, \mathcal{S}^{max} \subseteq \mathbb{M}^{[d]}$ such that $\mathcal{S}^{min}_1 = K = \mathcal{S}^{max}_1$, and what is the maximal constant $c \in (0,1)$*

such that $c\mathcal{S}^{max} \subseteq \mathcal{S}^{min}$*?* We have made first steps on this problem, but we still know very little. A powerful tool we plan to use is dilation of tuples to commuting normal tuples, which we now describe.

One of the key results of [46] is the construction, for every d-tuple $A \in M_n^d$ of contractive selfadjoint matrices, a commuting normal tuple of contractions $T \in B(H)^d$ which is a *dilation up to constant c for A* in the sense that

$$cA_i = V^* N_i V$$

for all i, where $V : \mathbb{C}^n \to H$ is an isometry. Equivalently, this can be rephrased as the fact that tuples lying within a certain matrix convex set (in this case the *matrix cube*) can be dilated to a normal tuple with prescribed spectrum (in this case, the inflation of the matrix cube by c^{-1}). We are working on a general method to construct dilations for noncommuting tuples in matrix convex sets to normal tuples with prescribed spectra. This will have consequences to inclusion problems of matrix convex sets, and in turn, this would show how one can "scale" a positive map to become a UCP map.

My original motivation for considering the interpolation problem is that any result on the existence and uniqueness of UCP maps between operator systems will have application to noncommutative Choquet theory [17, 18, 20]; for example the notion of hyperrigidity [21] is closely related to the *uniqueness* of extensions of UCP maps from a subsystem to a C*-algebra. Thus, complete understanding of the interpolation problem will be useful in situations where we seek hyperrigidity. In turn, understanding hyperrigidity may have consequences in other problems in operator algebras (see for example [53] for an application to *Arveson's essential normality conjecture*).

4. Conditions available for the research

The Faculty of Mathematics at the Technion can provide me with all the physical conditions required for carrying out the research. Currently there is a group in operator algebras consisting of two faculty members (B. Solel and I), a visiting professor (C. Schochet), a postdoc and a PhD student. Our group has ties with other groups in Israel, and we have meetings, at the local and at the national levels.

I request funding for a postdoc and a PhD student, so as to form a critical mass for studying the literature and carrying out research. All members of my group will need computer equipment, literature and travel funds. The funds will also be used to support the students and postdocs.

5. Pitfalls and alternative strategies

One "pitfall" is that I have not yet done much research in noncommutative function theory *per se*. I have already started studying the literature and attending conferences where there is a focus on nc function theory, and I will need to do more — e.g. seminars with my students on key papers. Much of the work in my previous ISF funded project (e.g. [37, 38, 32, 54, 56, 70]) can be viewed as noncommutative function theory on "commutative nc" varieties. In addition, I do have experience working on noncommutative operator algebras that fit the framework nc functions [37, 50, 71].

The above research project is ambitious and broad. I expect that together with strong postdocs and students, and perhaps also senior collaborators, I will be able to make significant progress on all of the problems. I am suggesting several different lines of research that can be carried out independently, so difficulties in one direction will not stop us from making progress in others.

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October 19, 2015

Israel Science Foundation
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43 Jabotinsky St.
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Jerusalem 91040
Israel

To whom it may concern:

I understand that Orr Shalit is applying to the ISF for funding, and that part of his program involves me as a collaborator. I hereby grant my consent that Orr Shalit include in his ISF proposal "Operator algebra geometry in noncommutative analysis" projects which we plan to carry out jointly.

Sincerely,

Ken Davidson

Kenneth R. Davidson, FRSC, FIF
University Professor

John E. McCarthy
Spencer T. Olin Professor of Mathematics
www.math.wustl.edu/~mccarthy

18/Oct/2015

To whom it may concern,

I hereby grant my consent that Orr Shalit include in his ISF proposal "Operator algebra geometry in noncommutative analysis" projects which we plan to carry out jointly.

Sincerely,

John McCarthy

John McCarthy

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Technion - Israel Institute of Technology
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Chair In Mathematics

הטכניון - מכון טכנולוגי לישראל
הפקולטה למתמטיקה

פרופ׳ ברוך סולל
הקתדרה למתמטיקה
ע״ש מרגרט מוסנפלדר הריס

October 2015

Letter of Consent

To whom it may concern,
I hereby grant my consent that Orr Shalit include in his ISF proposal "Operator algebra geometry in noncommutative analysis" projects which we plan to carry out jointly

Baruch Solel

Baruch Solel

Technion City, 32 000 Haifa, Israel

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קרית הטכניון, חיפה 32 000

Budget details

A. Personnel

Name (last, first)	Role in project	% time devoted	Salaries (in NIS)			
			1st year	2nd year	3rd year	4th year
Orr Moshe Shalit	PI	30	0	0	0	0
Postdoctoral fellow	RA/1	100	120,000	120,000	120,000	120,000
PhD student	RA/2	50	45,000	45,000	45,000	45,000
Master student	RA/3	50	40,000	40,000	40,000	40,000
Total Personnel			**205,000**	**205,000**	**205,000**	**205,000**

B. Supplies, Materials & Services

Item	Requested sums (in NIS)			
	1st year	2nd year	3rd year	4th year
laptop computer for PI	10,000	0	0	0
Participation in conferences (for student and postdoc)	6,000	6,000	6,000	6,000
laptop computer for research team	10,000	0	0	0
tablet for PI	4,000	0	0	0
Total supplies, materials & services	**30,000**	**6,000**	**6,000**	**6,000**

C. Miscellaneous

	Requested sums (in NIS)			
	1st year	2nd year	3rd year	4th year
Photocopies and office supplies	2,000	2,000	2,000	2,000
Publication charges in scientific journals	0	0	0	0
Professional literature	3,000	3,000	3,000	3,000
Internet Connection	1,500	1,500	1,500	1,500
Memberships in scientific associations	650	650	650	650
Total miscellaneous	**7,150**	**7,150**	**7,150**	**7,150**

Budget Summary

	Requested sums (in NIS)			
	1st year	2nd year	3rd year	4th year
Personnel, materials, supplies, services & miscellaneous	242,150	218,150	218,150	218,150
Overhead	41,166	37,086	37,086	37,086
Total budget	**283,316**	**255,236**	**255,236**	**255,236**
Annual Average	262,256			

Budget Justification:

Personnel:
Most of the budget is to cover the inevitable expenses of maintaining a research group (a postdoc and students).
Currently I have a postdoc (Dr. Eli Shamovich, who plans to continue working with me for the next two years) and a PhD student (Guy Salomon, who will continue working with me for the next three years) in my group. I hope to recruit also another masters student, and keep all positions filled throughout the duration of my grant.

The PhD student and the postdoc will help me work on the "complex" part of the project - the isomorphism problem for universal operator algebras and nc reproducing kernel Hilbert spaces.
The postdoc, who has a strong background in both operator algebras and algebraic geometry, will also contribute to the deeper problems such as the nc Nullstellensatz as well as the extension and approximation problems.
A masters student will help me work on the "real" part of the project, involving UCP maps and matrix convex sets, some part of which require less training to deal with.

Supplies and materials:
6000 NIS per year are requested to cover travel of the research assistants to conferences. This is completely necessary, and I would ask for twice the amount if it were possible.
I request funding to buy a laptop computer and a tablet to facilitate my work while traveling. Another laptop computer will be used by the research team.

Miscellaneous:
I request an additional modest amount to cover the costs of office supplies, membership in professional organisations, internet connection and literature.

Curriculum Vitae

Name: Shalit Orr Moshe

A. Academic Background

Date (from-to)	Institute	Degree	Area of specialization
2001-2003	Technion	BSc	Mathematics
2003-2005	Technion	MSc	Mathematics
2005-2009	Technion	PhD	Mathematics (Operator algebras and operator theory) Advisor: Prof. Baruch Solel

B. Previous Employment

Date (from-to)	Institute	Title	Research area
2014-present	Technion - Israel Institute of Technology	Assistant Professor	Mathematics
2011-2014	Ben Gurion University	Senior Lecturer	Mathematics
2009-2011	University of Waterloo	Postoctoral Fellow	Mathematics
2004-2006	Rafael Ltd.	Researcher	Computer vision/image processing
2002-2003	Rafael Ltd.	Researcher	Computer vision/image processing

C. Grants and Awards Received Within The Past Five Years

Date (from-to)	Research Topics	Funding Organization	Total (in NIS)
2012-2016	Operator Theory, Operator Algebras	Israel Science Foundation	680000
Comments			
2012-2016	Operator Theory, Operator Algebras	European Commission - Research Executive Agency	445000
Comments			
2013	Operator Algebras	German Israel Foundation	90000
Comments			
2015-2016	Operator algebras, completely positive maps	The Gerald Schwartz & Heather Reisman Foundation (Waterloo – Technion)	115000
Comments	Jointly with Prof. Ken Davidson (Waterloo) and Prof. Baruch Solel (Technion)		

O.M. SHALIT — LIST OF PUBLICATIONS

Chapters in books, surveys, expository

1. E. Levy and O.M. Shalit, *Dilation theory in finite dimensions: the possible, the impossible and the unknown*, Rocky Mountain J. Math. 44:1 (2014), 203–221.
2. G. Salomon and O.M. Shalit, *The isomorphism problem for complete Pick algebras: a survey*, to appear in *Proceedings of IWOTA 2014*, published as a volume in the series "Operator Theory: Advances and Applications", Springer.
3. O.M. Shalit, *A sneaky proof of the maximum modulus principle*, Amer. Math. Monthly 120:4 (2013), 359–362.
4. O.M. Shalit, *Operator theory and function theory in Drury-Arveson space and its qutients*, in "Hanbook of Operator Theory", ed. Daniel Alpay, Springer 2016.

Research papers in operator theory/operator algebras

1. K.R. Davidson, M. Hartz and O.M. Shalit, *Multipliers of embedded discs*, Complex Anal. Oper. Theory 9:2 (2015), 287–321.
2. K.R. Davidson, C. Ramsey and O.M. Shalit, *The isomorphism problem for some universal operator algebras*, Adv. Math. 228 (2011), 167–218.
3. K.R. Davidson, C. Ramsey and O.M. Shalit, *Operator algebras for analytic varieties*, Trans. Amer. Math. Soc. 367:2 (2015), 1121–1150.
4. M. Kennedy and O.M. Shalit, *Essential normality and the decomposability of algebraic varieties*, New York J. Math. 18 (2012), 877–890.
5. M. Kennedy and O.M. Shalit, *Essential normality, essential norms and hyperrigidity*, J. Funct. Anal. 268:10 (2015), 2990–3016.
6. M. Kerr, J.E. McCarthy and O.M. Shalit, *On the isomorphism question for complete Pick multiplier algebras*, Integral Equations Operator Theory 76:1 (2013), 39–53.
7. E. Levy and O.M. Shalit, *Continuous extension of a densely parametrized semigroup*, Semigroup Forum 78:2 (2009), 276–284.
8. D. Markiewicz and O.M. Shalit, *Continuity of CP-semigropus in the point-strong operator topology*, J. Operator Theory 64:1 (2010), 149–154.
9. J.E. McCarthy and O.M. Shalit, *Unitary N-dilations for tuples of commuting matrices*, Proc. Amer. Math. Soc. 14 (2013), 563–571.

10. S. Pal and O.M. Shalit, *Spectral sets and distinguished varieties in the symmetrized bidisc*, J. Funct. Anal. 266:9 (2014), 5779–5800.
11. O.M. Shalit, *E_0-dilations of strongly commuting CP_0-semigroups*, J. Funct. Anal. 255:1 (2008), 46–89.
12. O.M. Shalit, *What type of dynamics arise in E_0-dilations of commuting quantum Markov semigroups?*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 11:3 (2008), 393–403.
13. O.M. Shalit, *Representing a product system representation as a contractive semigroup and applications to regular isometric dilations*, Canad. Math. Bull. 53:3 (2010), 550–563.
14. O.M. Shalit, *Stable polynomial division and essential normality of graded Hilbert modules*, J. Lond. Math. Soc. 83:2 (2011), 273–289.
15. O.M. Shalit, *E-dilations of strongly commuting CP-semigroups (the nonunital case)*, Houston J. Math. 35:1 (2011), 203–232.
16. O.M. Shalit and M. Skeide, *Three commuting, unital, completely positive maps that have no minimal dilation*, Integral Equations Operator Theory 71:1 (2011), 55–63.
17. O.M. Shalit and B. Solel, *Subproduct systems*, Doc. Math. 14 (2009), 801–868.

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1. O.M. Shalit, *The overdeterminedness of a class of functional equations*, Aequationes Math. 74:3 (2007), 242–248.
2. O.M. Shalit, *Conjugacy of P-configurations and nonlinear solutions to a certain conditional Cauchy equation*, Banach J. Math. Anal. 3 (2009), 28–35.
3. O.M. Shalit, *Three remarks on a question of Aczél*, Aequationes Math. 84:3 (2012), 1–5.

Current ISF application No.: 195/16
PI Name: Orr Moshe Shalit
Research title: Operator-algebraic geometry in noncommutative analysis

Objectives, results, achievements & list of publications of a Recent ISF grant

Recent ISF grant No.: 474/12
Research title: Operator-algebraic geometry in the Drury-Arveson Hilbert module

Results and objectives

Let H_d^2 denote the Drury Arveson space in d variables, and let M_d denote its multiplier algebra. For an analytic variety V in the unit ball, let $M_V = M_d\big|_V$. The objectives of my ISF funded project (grant no. 474/12) were the following:

1. **The isomorphism problem for complete Pick algebras** - to study the structure of the algebras M_V in terms of the geometric structure of V ; specifically, to prove (under some conditions) that M_V and M_W are isomorphic if and only if V is the image of W under a biholomorphism.
2. To study the existence of an **effective Hilbert basis** for ideals in $\mathbb{C}[z_1,\dots z_d]$, that is, given a homogeneous ideal of polynomials I , to study whether it satisfies the **stable division property**, in the sense that there is a generating set $f_1,\dots,f_k$ for I such that every $h \in H$ can be written as $h = \sum g_i f_i$, with control on the norms of the terms, for example $\sum \|g_i f_i\| \leq C\|h\|$.
3. To study **Arveson's essential normality conjecture**, which asserts that the closure of every ideal $I \lhd \mathbb{C}[z_1,\dots z_d]$ in H_d^2 is an essentially normal module (with the module action obtained from the restriction of the shift). In particular, my aim was to study the problem for radical homogeneous ideals, in particular using an effective Hilbert basis theorem.

Problem 1 was the one on which I made the most progress, with the papers [2, 4, 5, 8, 10] directly resulting from this goal. We proved that M_V and M_W are isometrically isomorphic if and only if the varieties V and W are the images of one another under an automorphism of the ball [5]. We also established that if M_V and M_W are isomorphic then V is the image of W under a biholomorphism [5]. We obtained the converse direction in for special cases [2, 4], and one may say that the converse holds for the generic case of one dimensional algebraic varieties [2]. As a corollary, we obtained a sharpening of Henkin's theorem on extension of bounded analytic functions from a subvariety to the ball in the one dimensional case: every bounded analytic function on a "nice enough" one dimensional variety in the ball extends to a **multiplier** (in particular, to a bounded analytic function on the ball), with control on the norm.

On the other hand, we constructed counter examples showing that the converse does not hold generally [4] (that is, biholomorphism does not imply isomorphism). Moreover, we have geometric explanation for why some of these counter examples arise – it turns out that the pseudo hyperbolic metric on the variety V is a bi-Lipschitz invariant for the algebra M_V , that is, an isomorphism between the algebras must induce a bi-Lipschitz map between the varieties when they are given the pseudo hyperbolic metric. Moreover, there were many other discoveries on operator theory in the unit ball made along the way, for example a Hopf type lemma for embedded discs obtained in [4] (a nicely embedded disc in the ball must meet the boundary transversally). Another surprising discovery was the connection with Dirichlet series detailed in [10] – we found that the Drury-Arveson space in any dimension (including infinity) can be represented as a space of Dirichlet series on a half plane; in other words, we discovered universal complete Pick algebras in **one complex variable**! Although there are still many open questions, the first part of the project can be described as a complete success.

On problem 2 I have made less progress than I hoped, and the progress I made was not in the direction I supposed to make progress. Moreover, my idea to use techniques from harmonic analysis to attack this problem was perhaps naïve – I know now that some of the techniques I wished to apply (a paper of Costea, Sawyer and Wick) are not applicable and it is beyond my abilities to adapt them. However, other new techniques coming from harmonic analysis (papers of Fang and Xia) **were** applicable, and I used them to obtain the main results of [9]. In [9] we showed that if an ideal (not necessarily homogeneous) has the **approximate stable division property** then it is essentially normal. Moreover, we proved that quasi-homogeneous ideals in two variables have the (approximate) stable division property, so they are all essentially normal.

I worked successfully also on problem 3, but mostly not by the main approach that I suggested. The paper [9] contains results on essential normality using approximate stable division – which is along the lines suggested in my proposal - but the papers [1, 6] use completely different methods. The paper [1] contains the result that the closure of a radical ideal corresponding to a union of linear subspaces is essentially normal; this result has not been obtained by any other method. The paper [6] is the most interesting paper I have on this problem, introducing a completely new point of view that relates the essential normality problem to noncommutative Choquet theory. It shows that the shift acting on the quotient of H^2_d by a homogeneous ideal is essentially normal if and only if it is **hyperrigid** in the sense of Arveson. My student Guy Salomon is now working on applying this result to obtain new proofs of essential normality of ideals and quotient modules.

Studying the literature on Drury-Arveson space as the first part of my project, I became a big enough expert that I wrote chapter devoted to this space in the Handbook of Operator Theory [7].

Finally, my work naturally expanded to close areas not strictly covered in my original proposal: the paper [3] contains results on the connection between operator theory and the geometry of the symmetrized bidisc (this is similar to the theme of my proposal, but on a domain other than the unit ball); the paper [11] contains (along many other things) noncommutative analogues of the commutative questions my proposal raised, and is perhaps a first step in the direction of my current proposal.

List of publications

1. M. Kennedy and O.M. Shalit. Essential normality and the decomposability of algebraic varieties, *New York J. Math.* 18 (2012), 877-890.
2. M. Kerr, J.E. McCarthy and O.M. Shalit. On the isomorphism question for complete Pick multiplier algebras, *Integral Equations Operator Theory* Vol 76:1 (2013), 39-53.
3. S. Pal and O.M. Shalit. Spectral sets and distinguished varieties in the symmetrized bidisc, *J. Funct. Anal.* Vol. 266:9 (2014), 5779-5800.
4. K.R. Davidson, M. Hartz and O.M. Shalit. Multipliers of embedded disks, *Complex Analysis Operator Theory,* Vol. 9 (2015), 287-321.
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6. M. Kennedy and O.M. Shalit. Essential normality, essential norms and hyperrigidity, *J. Funct. Anal.* Vol. 268:10 (2015), 2990-3016.
7. O.M. Shalit. Operator theory and function theory in Drury-Arveson space and its quotients, (chapter in book) in *Handbook of Operator Theory*, editor Daniel Alpay, Springer (2016).
8. G. Salomon and O.M. Shalit. The isomorphism problem for complete Pick algebras: a survey, prepared for Proceedings of IWOTA 2014, to appear in the series *Operator Theory: Advances and Applications.*
9. S. Biswas and O.M. Shalit. Stable division and essential normality: the non-homogeneous and quasi homogeneous cases, submitted.
10. J.E. McCarthy and O.M. Shalit. Spaces of Dirichlet series with the complete Pick property, submitted for publication.
11. E. Kakariadis and O.M. Shalit. On operator algebras associated with monomial ideals in noncommuting variables, submitted for publication.