RESEARCH STATEMENT - ORR MOSHE SHALIT

This document describes my research, emphasizing the accomplishments achieved since 2017.

1. Dilation theory, matrix convexity and operator systems

1.1. **The connection between matrix ranges and the structure of operator systems.** Since Arveson's seminal papers [7, 8], which initiated the theory of operator spaces, operator systems and the systematic study of nonselfadjoint operator algebras, completely positive (CP) maps have played an important role in the theory of operator algebras and in operator theory. Several years ago I was led to consider the connection between CP maps and matrix convexity. The interplay of CP maps, dilation theory, matrix convexity and operator algebras has become the most important theme of my research in the last few years (see also my survey [71]).

For d-tuple A, we defined in [24] its *matrix range* to be the free set

$$\mathcal{W}(A) = \cup_n \mathcal{W}_n(A) = \cup_n \{\psi(A) : \psi : C^*(A) \to M_n \text{ is UCP}\}.$$

This notion is an organic extension of Arveson's notion of matrix range of a single element [8], and following Arveson we obtained the following basic theorem.

Theorem 1 (Davidson, Dor-On, Shalit and Solel [24])**.** *There exists a UCP map sending A_i to B_i for all $i = 1, \ldots, d$ if and only if $\mathcal{W}(B) \subseteq \mathcal{W}(A)$. There exists a unital completely isometric map sending A_i to B_i for all $i = 1, \ldots, d$ if and only if $\mathcal{W}(B) = \mathcal{W}(A)$.*

From Theorem 1 we recovered earlier interpolation results as special cases; Helton, Klep and McCullough's result that the operator space structure is encoded in the free spectrahedron [43], as well as the result of Li and Poon that a normal tuple B is the image under a UCP map of a normal tuple A if and only if the joint spectrum of B is contained in the convex hull of the joint spectrum of A [54]. In both cases, we extended the results from matrices to operators on infinite dimensions.

By Theorem 1 the matrix range determines the operator system $\operatorname{span}\{1, A_1, A_1^*, \ldots, A_d, A_d^*\}$ of A. To what extent does $\mathcal{W}(A)$ determine the structure of A? Partial answers were obtained in [24].

Theorem 2 (Davidson, Dor-On, Shalit and Solel [24])**.** *Let A and B be d-tuples of operators such that $C^*_{env}(A) = C^*(A)$, $C^*_{env}(B) = C^*(B)$, and $C^*(A)$ and $C^*(B)$ contain no nonzero compact operator. Then $\mathcal{W}(B) = \mathcal{W}(A)$ if and only if A and B are approximately unitarily equivalent.*

The opposite extreme of tuples of compact operators was also studied in [24], and a complete result for compacts was obtained later by my postdoc Ben Passer and me [61].

Theorem 3 (Passer and Shalit [61])**.** *Let A and B be two fully compressed d-tuples of compact operators. Then $\mathcal{W}(A) = \mathcal{W}(B)$ if and only if A is unitarily equivalent to B.*

Fully compressed means that the tuple cannot be compressed to a tuple with the same matrix range. This condition can be understood in terms of the noncommutative Shilov boundary [7].

Theorem 4 (Passer and Shalit [61])**.** *For a compact tuple A the following are equivalent.*

1. *A is fully compressed.*
2. *A is multiplicity-free, and the Shilov ideal of S_A in $C^*(S_A)$ is trivial.*
3. *A is minimal and nonsingular.*

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The proofs involved understanding the relationship between Arveson's boundary representations and various sorts of *extreme points* in the matrix range of a tuple. We have some results about the latter topic as well. We expected to extend Theorem 3 to GCR tuples, but Davidson and Passer later built on our ideas and proved that the theorem actually extends to general tuples [26].

1.2. **Random matrix ranges.** Matrix ranges being important, one is led to ask a variety of questions about them. Together with Malte Gerhold, my postdoc, we asked *what does a "typical" random matrix range look like*? In [35] we pioneered the study of random matrix ranges. We studied the relationship between various modes of convergence for tuples of operators and continuity of matrix ranges with respect to the Hausdorff metric. We proved an effective version of the Effros-Winkler Hahn-Banach type separation theorem for matrix convex sets, and applied it to show that the matrix range of a tuple generating a continuous field of C*-algebras is levelwise continuous in the Hausdorff metric. Using known results on strong convergence of matrix ensembles we identified the sets to which the matrix ranges of independent Wigner or Haar ensembles converge.

Theorem 5 (Gerhold and Shalit [35])**.** *Let $X^N = (X_1^N, \ldots, X_d^N)$ be a Wigner ensemble and assume that $\mathbb{E}(|(X_k^N)_{ij}|^4) < \infty$ for all i, j. Let $s = (s_1, \ldots, s_d)$ be the semicircular d-tuple. Then $\mathcal{W}(X^N)$ converges levelwise in the Hausdorff metric to $\mathcal{W}(s)$ almost surely, that is, for all n,*

$$\lim_{N\to\infty} \mathrm{d_H}(\mathcal{W}_n(X^N), \mathcal{W}_n(s)) = 0 \ , \ a.s.$$

Theorem 6 (Gerhold and Shalit [35])**.** *Let $U^N = (U_1^N, \ldots, U_d^N)$ be an ensemble of d independent $N \times N$ unitaries distributed according to the Haar measure in $\mathcal{U}_N$ and let $u = (u_1, \ldots, u_d)$ be free Haar unitaries. Then $\mathcal{W}(U^N)$ converges levelwise in the Hausdorff metric to $\mathcal{W}(u)$ almost surely,*

$$\lim_{N\to\infty} \mathrm{d_H}\left(\mathcal{W}_n(U^N), \mathcal{W}_n(u)\right) = 0 \ , \ a.s.$$

For the case $n = 1$ and $d = 2$, Theorem 5 describes the first level of the matrix range of two selfadjoint matrices and we obtain that, for matrices drawn from, say, the Ginibre ensemble, the numerical range converges to a disc (as $N \to \infty$), recovering a result of Collins et al. [23].

1.3. **Minimal and maximal matrix convex sets and dilations.** In the ground breaking paper [44] Helton, McCullough, Klep and Schweighofer reformulate Ben-Tal and Nemirovski's relaxation to the NP hard *matrix cube problem* of determining whether a spectrahedron contains the unit cube [17], and they find the optimal constant for this relaxation. A key in their analysis is the simultaneous dilation of *all* self adjoint contractions on a finite dimensional space to a *commuting family* of selfadjoints on a larger space, with spectrum confined to some cube. These results, combined with Theorem 1 and its consequences, led us to investigate inclusion problems for matrix convex sets with an emphasis on the dilation theoretic perspective.

If $A = (A_1, \ldots, A_d) \in B(H)^d$ and $B = (B_1, \ldots, B_d) \in B(K)^d$ where $H \subset K$, then we say that B is a *dilation* of A, and that A is a *compression* of B, if $A_i = P_H B_i\big|_H$ for $i = 1, \ldots, d$, where P_H denotes the orthogonal projection of K onto H. If B is a dilation of A, we write $A \prec B$. The groundbreaking dilation result of [44] can be paraphrased as follows: *There exists a constant $\vartheta(n)$ such that for any Hilbert space H of finite dimension n, and every d-tuple $A \in B(H)^d$ of selfadjoint contractions, there exists a Hilbert space $K \supset H$ and a d-tuple $N \in B(K)^d$ of commuting selfadjoint contractions such that*

$$A \prec \vartheta(n) B. \tag{1}$$

Significantly, [44] contains sharp information about the optimal constant $\vartheta(n)$ and it is shown that $\vartheta(n) \sim \sqrt{n}$. This is strikingly different from classical dilation theorems in operator theory, where one dilates commuting tuples to better understood commuting operators. Here, one dilates noncommuting selfadjoint operators to *commuting* ones, at the price of a scale factor.

A *matrix convex set* is a free set $\mathcal{S} = \cup_{n\geq 1}\mathcal{S}_n$, where each $\mathcal{S}_n$ is a set of d-tuples of $n\times n$ matrices, that is invariant under *matrix convex combinations.* Inspired by [44] we ask: *if $\mathcal{S}, \mathcal{T}$ are two matrix convex sets and $\mathcal{S}_1 \subseteq \mathcal{T}_1$, what can we say about the inclusion of $\mathcal{S}$ in $\mathcal{T}$?* To answer this question we studied the minimal and maximal matrix convex sets $\mathcal{W}^{min}(K)$ and $\mathcal{W}^{max}(K)$ over a set $K \subseteq \mathbb{R}^d$.

One of the most useful contributions we made in [24] is the following observation:

$$\mathcal{W}^{min}(K) = \{X : \exists N \text{ normal s.t. } \sigma(N) \subseteq K \text{ and } N \text{ is a dilation of } X\}, \tag{2}$$

where $\sigma(N)$ denotes the joint spectrum of the normal (commuting) tuple N.

In [62], written with my postdoc Ben Passer and with Baruch Solel, we introduced the *dilation constant* $\theta(K)$ for a compact convex set K, which is defined to be the best constant $\theta(K)$ such that

$$\mathcal{W}^{\max}(K) \subseteq \theta(K)\mathcal{W}^{\min}(K).$$

I will now present two of our main results.

Theorem 7 (Passer, Shalit and Solel [62])**.** *$\theta(K) = 1$ if and only if K is a simplex.*

By Theorem 7 the only convex sets $K \subset \mathbb{R}^d$ over which there is only one matrix convex set, are simplices. This improves a result of Fritz, Netzer and Thom who proved this for K a polytope [31]. We showed that it suffices to check $\mathcal{W}^{\max}_n(K) = \mathcal{W}^{\min}_n(K)$ only for $n = 2^{d-1}$, or even $n = 2$ for "simplex pointed" sets. We asked whether equality $\mathcal{W}^{\max}_n(K) = \mathcal{W}^{\min}_n(K)$ at $n = 2$ already implies K is a simplex. Huber and Netzer gave a positive answer in the case that K is a polytope, remarking that "the difference [between the minimal and maximal matrix convex set over K] can always be seen at the first level of non-commutativity, i.e. for matrices of size 2" [46]. The problem was resolved in full by Auburn et al., who showed how the positive solution of our problem follows form their resolution of a conjecture of Barker, that the minimal and maximal tensor products of two finite-dimensional proper cones coincide if and only if one of the two cones is "classical" [12].

Theorem 8 (Passer, Shalit and Solel [62])**.** *Let $\overline{\mathbb{B}}_{p,d}$ be the unit ball in $\mathbb{R}^d$ with the ℓ^p norm. Then*

$$\theta(\overline{\mathbb{B}}_{p,d}) = d^{1-|1/2-1/p|}.$$

For $p \neq 2$ the sharp constant $\theta(\overline{\mathbb{B}}_{p,d}) = d^{1-|1/2-1/p|}$ was new ($d = 2$ was known, e.g., [24, 31]). The cases $p = 1, \infty$ are of special interest, and were cited for estimates regarding joint measurability of quantum effects in quantum information theory [20, 21]. The case $p = \infty$ means that

$$\mathcal{W}^{\max}([-1,1]^d) \subseteq \sqrt{d}\mathcal{W}^{\min}([-1,1]^d). \tag{3}$$

and that $\sqrt{d}$ is the optimal constant; by our observation (2) this can be interpreted as follows.

Theorem 9 (Passer, Shalit and Solel [62])**.** *For every d-tuple A of selfadjoint contractions, there exists a commuting d-tuple of selfadjoint contractions B, such that*

$$A \prec \sqrt{d}B.$$

Note that in Theorem 9 the constant depends on d but not on the dimension of H, whereas in the inequality (1) the constant depended on $\dim H$ but not on d. The paper [62] has several other results of comparable sharpness, for example we find the constants for unit balls of ℓ^p spaces over the complex numbers, and study *minimal dilation hulls.* Our work inspired additional interesting research not mentioned above, for example the interesting [60].

1.4. **Dilation constants and applications of dilation techniques to noncommutative settings.** Theorem 9 begs the natural question: *what is the smallest constant c such that for every d-tuple of contractions A, there exists a commuting normal tuple N of contractions such that $A \prec cN$?* Let C_d be the optimal constant. By Theorem 9, if A is selfadjoint then $c = \sqrt{d}$ works, and sharpness of the constant gives $C_d \geq \sqrt{d}$. Passer obtained $C_d \leq \sqrt{2d}$ [59]. The constant C_d is the optimal scale c such that the following von Neumann type inequality holds

$$\|p(A)\| \leq \sup\{\|p(z)\| : z \in c\overline{\mathbb{D}}^d\} \tag{4}$$

for every tuple of contractions A and every matrix valued polynomial of degree ≤ 1. One can consider C_d to be a fundamental "constant of nature" pertaining to the structure of operator systems, and thus it is worth pursuing.

Given two d-tuples A and B, we extend the notation $A \prec B$ to mean that A is a compression of a $*$-isomorphic copy of B; equivalently, there exist a UCP map such that $B_i \mapsto A_i$ for all $i = 1, \ldots, d$. We let $c(A, B)$ be the minimal constant c such that $A \prec cB$. Note that

$$C_d = \max\{c(A, Z) : A \text{ is a tuple of contractions}\} = \max\{c(U, Z) : U \text{ is a tuple of unitaries}\}$$

where $Z = (Z_1, \ldots, Z_d)$ is the d-tuple of coordinate functions in $C(\overline{\mathbb{D}}^d)$. This suggests that to obtain lower bounds for C_d we should attempt to calculate $C(U, Z)$ for particular tuples of unitaries U. This line of reasoning led to my paper [34], which was written jointly with my postdoc Malte Gerhold. In our quest to find C_d, we considered the problem of finding the constants $c_\theta = c((u_\theta, v_\theta), (u_0, v_0))$, where (u_θ, v_θ) is the universal pair of unitaries that satisfy $v_\theta u_\theta = e^{i\theta} u_\theta v_\theta$, i.e., the generators of the rotation algebras. The pair (u_0, v_0) is the universal pair of commuting unitaries, so is the same as $Z = (Z_1, Z_2)$ from above. In [34] we computed the sharp constant c_θ.

Theorem 10 (Gerhold and Shalit [34])**.**

$$c_\theta = \frac{4}{\|u_\theta + u_\theta^* + v_\theta + v_\theta^*\|}. \tag{5}$$

It is interesting to note that the operator $h_\theta = u_\theta + u_\theta^* + v_\theta + v_\theta^*$ which appears in (6) is a so-called *almost Mathieu operator* which is the subject of intensive research in mathematical physics and is the Schrödinger operator appearing in the context of Hosftadter's Butterfly [45]. Determination of the precise value of the norm $\|h_\theta\|$ for all θ is difficult, but using a mix of numerical and analytical considerations we obtained the lower bound $C_2 \geq \max_\theta c_\theta \approx 1.54 > \sqrt{2}$, the best known to date.

The natural symmetry that the family of rotation algebra enjoys led us to dilate θ-commuting unitaries to θ'-commuting unitaries, and we computed constants $c_{\theta,\theta'} = c((u_\theta, v_\theta), (u_{\theta'}, v_{\theta'}))$. We found the precise value $c_{\theta,\theta'} = c_{\theta-\theta'}$. Using different methods, we also found in [34] the bound

$$c_{\theta,\theta'} \leq e^{\frac{1}{4}|\theta-\theta'|}. \tag{6}$$

This has surprising applications: new proofs of the Lipschitz continuity of the norm $\theta \mapsto \|h_\theta\|$ [16] and the 1/2-Hölder continuity of the spectrum $\theta \mapsto \sigma(h_\theta)$ [13].

The exciting results and applications in [34] were quickly subsumed by the paper [33], written with Malte Gerhold, Satish Pandey (my postdoc), and Baruch Solel. We studied d-tuples of unitaries $u = (u_1, \ldots, u_d)$ using dilation theory and matrix ranges. Given two such d-tuples u and v, we seek the minimal dilation constant $c = c(u, v)$ such that $u \prec cv$. We define the *dilation distance*

$$\mathrm{d_D}(u, v) = \log \max\{c(u, v), c(v, u)\} \tag{7}$$

on the set $\mathcal{U}(d)$ of equivalence classes of $*$-isomorphic d-tuples of unitaries. Here, we say that u is equivalent to u', and we write $u \sim u'$, if u' is some $*$-isomorphic image of u. We compared the dilation distance to what we refer to as the *Haagerup-Rørdam distance* $\mathrm{d_{HR}}$ determined by

$$\mathrm{d_{HR}}(u, v) = \inf\left\{\|u' - v'\| : u', v' \in B(\mathcal{H})^d, u' \sim u \text{ and } v' \sim v\right\}. \tag{8}$$

The following inequality connects dilation theory and representations in a precise analytic manner.

Theorem 11 (Gerhold, Pandey, Shalit and Solel [33])**.** *There is a constant $K \leq 56$ such that for all $u, v \in \mathcal{U}(d)$*

$$\mathrm{d_{HR}}(u,v) \leq K\,\mathrm{d_D}(u,v)^{1/2}. \tag{9}$$

We also define the *matrix range distance* to be the Hausdorff distance between matrix ranges

$$\mathrm{d_{mr}}(u,v) := \sup_n \mathrm{d_H}\big(\mathcal{W}_n(u), \mathcal{W}_n(v)\big).$$

We show $\mathrm{d}_{\mathrm{m}r} \leq \mathrm{d}_{\mathrm{H}R}$ and $\mathrm{d}_{\mathrm{m}r} \leq 2\,\mathrm{d_D}$. As unitary tuples are hyperrigid [11], Theorem 1 implies that $\mathrm{d_{mr}}$ is a metric, thus so are the other two distances. The metrics are equivalent on the set $\mathcal{U}_0(d)$ of unitary d-tuples whose matrix range contains some neighborhood of the origin.

For particular classes of unitary tuples we found explicit bounds for the dilation constant. For example, if for a real antisymmetric $d \times d$ matrix $\Theta = (\theta_{k,\ell})$ we let u_Θ be the universal unitary tuple $(u_1, \ldots, u_d)$ satisfying $u_\ell u_k = e^{i\theta_{k,\ell}} u_k u_\ell$. We found that

$$c(u_\Theta, u_{\Theta'}) \leq e^{\frac{1}{4}\|\Theta - \Theta'\|},$$

generalizing (6). However, we get much more, because by Theorem 11, this shows that the map $\Theta \mapsto u_\Theta$ is $1/2$-Hölder continuous into the space $\mathcal{U}(d)$ of equivalence classes of unitary d-tuples. Inspired by Haagerup and Rørdam, we proved that for every α-Hölder continuous path in $\mathcal{U}(d)$ with respect to $\mathrm{d_{HR}}$ can be lifted to a path that is α-Hölder continuous with respect to the operator norm from the interval into the group of unitaries on some Hilbert space.

Putting everything together, we recovered by our dilation theoretical techniques the result of Haagerup-Rørdam [38] (in the $d = 2$ case) and Gao [32] (in the $d \geq 2$ case) that the noncommutative tori form a continuous field of C*-algebras in the very strong sense, that there exists a map $\Theta \mapsto U(\Theta) \in B(H)^d$ such that $U(\Theta) \sim u_\Theta$ and

$$\|U(\Theta) - U(\Theta')\| \leq K\|\Theta - \Theta'\|^{1/2}.$$

In the case $d = 3$ we used a mix of numerical and analytical considerations to get the new lower bound $C_3 \geq 1.858$, improving on the previously known lower bound $C_3 \geq \sqrt{3}$.

In [33] we also considered the universal d-tuple of noncommuting unitaries u, the d-tuple of free Haar unitaries u_f, and the universal d-tuple of commuting unitaries u_0.

Theorem 12 (Gerhold, Pandey, Shalit and Solel [33])**.**

$$c(\mathrm{u}, u_f) = c(u_0, u_f) = \frac{d}{\sqrt{2d-1}}$$

and

$$2\sqrt{1 - \frac{1}{d}} \leq c(u_f, u_0) \leq 2\sqrt{1 - \frac{1}{2d}}.$$

Remarkably, we have the bound $c(u_f, u_0) \leq 2$ independently of d. Combining Theorem 12 with the observations that $c(\mathrm{u}, u_0) \leq c(\mathrm{u}, u_f)c(u_f, u_0)$ and that $C_d = c(\mathrm{u}, u_0)$, we recover Passer's upper bound $C_d \leq \sqrt{2d}$ [59] with a different proof, leaving open whether this upper bound is sharp.

1.5. **Bounded perturbation of the Heisenberg commutation relation.** In the previous subsection we described how our dilation methods recovered Haagerup and Rørdam's $1/2$-Hölder norm-continuous path $\theta \mapsto (U_\theta, V_\theta)$, where U_θ, V_θ are all unitaries on the same Hilbert space such that $V_\theta U_\theta = e^{i\theta} U_\theta V_\theta$. In the original paper [38] this was proved via a result of independent interest on bounded perturbations of unbounded operators. Our proof avoids unbounded operators, and this gave us some satisfaction that the dilation method that we discovered is quite powerful. However, we got curious whether we could use dilations not only to bypass the result on unbounded operators, but to recover and generalize it as well. This is how the paper [36] was born.

A pair of selfadjoint (unbounded) operators P and Q is said to satisfy the *Heisenberg commutation relation* if the corresponding unitary groups $u(t) = e^{itP}$ and $v(t) = e^{itQ}$ satisfy

$$u(s)v(t) = e^{ist}v(t)u(s),$$

a condition which is customarily interpreted as

$$[P,Q] = -iI.$$

It is well known that that there exists a unique representation of the Heisenberg commutation relation that is irreducible, namely: the representation on $L^2(\mathbb{R})$ in which $P = -i\frac{\mathrm{d}}{\mathrm{d}x}$ is the momentum operator and $Q = M_x$ (where $(M_x f)(x) = xf(x)$) is the position operator, the canonical pair of operators from basic quantum mechanics.

A natural question raised by von Neumann [76] was this: *can one approximate somehow P and Q by a pair of commuting selfadjoint operators?* It is not entirely clear what precisely he had in mind, but it is by now folklore that one *cannot* approximate the canonical P and Q in norm with a pair of strongly commuting P_0 and Q_0 (*strongly commuting* means, essentially, that the spectral measures of P_0 and Q_0 commute). There is a natural index obstruction, and this obstruction remains for finite multiplicity versions of P and Q. However, Haagerup and Rørdam solved the problem for the infinite ampliations of P and Q. A higher dimensional generalization of this was obtained by Gao [32]. We recovered Gao's result, improving the bounds.

Theorem 13 (Haagerup and Rørdam [38], Gao [32], Gerhold and Shalit [36])**.** *Let $d = 2n$ and let Θ be a real nonsingular antisymmetric $d \times d$ matrix. Let $P_1, \ldots, P_d$ be the generators of one-parameter unitary groups $u_1, \ldots, u_d$ that commute according to Θ. For any real antisymmetric $d\times d$ matrix Θ', the infinite ampliation $P^\infty := P \otimes 1_{\ell^2(\mathbb{N})}$ of P is a bounded perturbation of a d-tuple Q of selfadjoint operators that generate d unitary groups that commute according to Θ' such that*

$$\|P_k^\infty - Q_k\| \leq \tfrac{5}{\sqrt{2}}\|\Theta - \Theta'\|^{1/2}$$

for all $k = 1, \ldots, d$.

Letting $d = 2$ and $\theta_{12} = -\theta_{21} = 1$ and $\Theta' = 0$ we recover Haagerup and Rørdam's result with the somewhat better bound $\frac{5}{\sqrt{2}} \approx 3.54$ compared to their best estimate $\sqrt{45} \approx 6.71$.

To prove the theorem, we first generalize the dilation distance (7) and the HR distance (8) from tuples of unitaries to tuples of one-parameter unitary groups. We prove the appropriate analogue of Theorem 11 for unitary groups, showing that the dilation distance dominates the distance between representations of the groups. We then observe that two unitary groups a finite distance from each other have generators that are a bounded perturbation one of the other. Then, we construct a concrete dilation of unitary groups commuting according to Θ and Θ' by making use of the Weyl representation. Putting everything together gives Theorem 13.

2. Multivariable operator theory and algebras of (noncommutative) functions

2.1. Background on the isomorphism problem for complete Pick algebras. The *Drury-Arveson space* H^2_d is the reproducing kernel Hilbert space on the unit ball $\mathbb{B}_d \subset \mathbb{C}^d$ determined by the kernel $k(z,w) = \frac{1}{1-\langle z,w\rangle}$ (see my survey [70]). Let $\mathcal{M}_d$ be the multiplier algebra of H^2_d,

$$\mathcal{M}_d = \{f : \mathbb{B}_d \to \mathbb{C} : fh \in H^2_d \text{ for all } h \in H^2_d\} \subseteq H^\infty(\mathbb{B}_d).$$

By *variety* we shall refer to a zero set of multipliers. For a variety $V \subseteq \mathbb{B}_d$, we denote

$$\mathcal{M}_V = \{f\big|_V : f \in \mathcal{M}_d\}.$$

Every irreducible complete Pick multiplier algebra is of the form $\mathcal{M}_V$ for some V [1]. The *isomorphism problem* is the problem whether the "geometry" of V is a complete invariant for the "structure" of $\mathcal{M}_V$ (see my survey [64]).

Theorem 14 (Davidson, Ramsey and Shalit [29]). *Let $V, W \subseteq \mathbb{B}_d$ be two varieties. Then $\mathcal{M}_V$ and $\mathcal{M}_W$ are isometrically isomorphic, if and only if there is a biholomorphic automorphism of the ball that maps V onto W.*

With regards to algebraic isomorphism, one implication holds in considerable generality.

Theorem 15 (Davidson, Ramsey and Shalit [29]). *Let $V, W \subseteq \mathbb{B}_d$ be two varieties that are a union of a discrete variety and a finite union of irreducible varieties. If $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic, then V and W are multiplier biholomorphic.*

Here *multiplier biholomorphic* means that the biholomorphism and its inverse are vector valued multipliers. Michael Hartz, Ken Davidson and I showed that the converse fails in general [25]. In fact, we observed that *multiplier biholomorphism is not an equivalence relation.* Seeking a complete invariant for these algebras, we showed that if $\mathcal{M}_V \cong \mathcal{M}_W$ then V and W are bi-Lipschitz w.r.t. the pseudohyperbolic metric in the ball. But this too isn't a complete invariant [25]. However, a converse to Theorem 15 holds within certain classes of varieties, e.g., two homogeneous varieties V, W are biholomorphic if and only if $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic. We proved this in [28] with Chris Ramsey and Ken Davidson under some assumptions, which were later removed by Hartz [39].

The isomorphism problem was also resolved for V, W one dimensional and sufficiently nice — by Alpay, Putinar and Vinnikov for discs [5], by Arcozzi, Rochberg and Sawyer for planar domains [6], and by Kerr, McCarthy and me for finite Riemann surfaces [53]. All cases assume V, W are images of proper holomorphic maps with finitely many ramification points that extend injectively to regular C^2 functions on the boundary (early versions required a transversality condition at the boundary, but in [25] we proved a Hopf type lemma showing that one dimensional varieties as above always meet the boundary transversally). Specifically, we show that if V is as above, then $\mathcal{M}_V = H^\infty(V)$, with comparable norms. It follows that $\mathcal{M}_V$ is isomorphic to $\mathcal{M}_W$ if and only if V and W are biholomorphic. Another striking corollary to the above theorem is a Henkin type extension result: for $V \subset \mathbb{B}_d$ as above, there exists a constant $C > 0$, such that for every $f \in H^\infty(V)$, there exists a multiplier $F \in \mathcal{M}_d \subset H^\infty(\mathbb{B}_d)$ such that $F\big|_V = f$ and $\|F\|_\infty \leq \|F\|_{mult} \leq C\|f\|_\infty$ [53].

2.2. **A quantitative approach to the isomorphism problem.** Consider a finite set $X \subset \mathbb{B}_d$, and construct the quotient module $\mathcal{H}_X := \overline{\operatorname{span}}\{k_\lambda : x \in X\} = H^2_d\big|_X$ together with its multiplier algebra $\mathcal{M}_X = \operatorname{Mult}(\mathcal{H}_X) = \mathcal{M}_d\big|_X$. We know from the previous subsection that if $Y \subset \mathbb{B}_d$, then $\mathcal{M}_X$ and $\mathcal{M}_Y$ are isometrically isomorphic if and only if Y is the image of X under an automorphism of the ball. But what if Y is very close to X under an automorphism? Are the algebras then in some sense "almost" isometrically isomorphic? More interestingly, if the algebras are "almost" isometrically isomorphic, does this mean that the sets are close to being an automorphic image one of the other? In the [57], my undergraduate supervisee Danny Ofek, my postdoc Satish Pandey and I introduced a quantitative version of the isomorphism problem that treats this problem.

Let ρ_{ph} be the pseudohyperbolic metric on the unit ball $\mathbb{B}_d$. This metric induces the Hausdorff metric ρ_H on subsets of the ball, which in turn gives rise to an automorphism invariant Hausdorff distance between subsets

$$\tilde{\rho}_H(X, Y) = \inf\{\rho_H(X, \Phi(Y)) : \Phi \in \operatorname{Aut}(\mathbb{B}_d)\}.$$

We introduced a distance function ρ_{RK} which is analogous to the Banach-Mazur distance

$$\rho_{RK}(\mathcal{H}_1, \mathcal{H}_2) = \log\left(\inf\left\{\|T\|\|T^{-1}\| : T : \mathcal{H}_1 \to \mathcal{H}_2 \text{ is an RKHS isomorphism}\right\}\right).$$

This distance function quantifies how far two Hilbert function spaces are from being isometrically isomorphic as reproducing kernel Hilbert spaces. We also introduced a distance function ρ_M, defined via a similar formula, that quantifies how far two multiplier algebras are from being completely isometrically isomorphic. The main results in [57] are that for two finite subsets X and Y of the unit ball, $\tilde{\rho}_H(X, Y)$ is small if and only if $\rho_{RK}(\mathcal{H}_X, \mathcal{H}_Y)$ is small, and this happens if and only if

$\rho_M(\mathcal{M}_X, \mathcal{M}_Y)$ is small. When one of the distances is zero this is what Theorem 14 says. It is worth mentioning that Ofek and Sofer found many natural weighted Hardy spaces for which this type of theorem fails [58], in fact already the qualitative Theorem 14 fails.

2.3. **Classification of algebras of bounded noncommutative functions on subvarieties of the noncommutative unit ball.** Noncommutative (nc) function theory has flourished during the last 15 years [2, 3, 14, 15, 42, 48]. Several years ago I realized that many of the operator algebras that I and others studied could be realized as algebras of bounded nc function on subvarieties of the nc unit ball. In particular, the isomorphism problem discussed above can be viewed as as a special case of a wider problem that takes place in the noncommutative world.

Let M_n^d be the set of all d-tuples of $n \times n$ such matrices, and put $\mathbb{M}^d = \cup_{n=1}^\infty M_n^d$. A subset $\Omega \subseteq \mathbb{M}^d$ is called a *free set*. A free set Ω is said to be a *nc set* if it is closed under direct sums. For a nc set Ω we denote $\Omega_n = \Omega \cap M_n^d$, and $\Omega = \cup_{n=1}^\infty \Omega_n$. The *similarity envelope* of a nc set Ω is the nc set $\widetilde{\Omega}$ consisting of all tuples $S^{-1}XS$ jointly similar to some $X \in \Omega$. We say that a nc set Ω is a *nc domain* if Ω_n is open in M_n^d for all n. Our central example of a nc domain is the *nc unit ball*

$$\mathfrak{B}_d = \{X \in \mathbb{M}^d : \| \sum X_j X_j^* \| < 1\}.$$

A function f from a nc set $\Omega \subseteq \mathbb{M}^d$ to $\mathbb{M}^1$ is said to be a *nc function* if it is graded, respects direct sums, and respects similarities. Free polynomials are the most important example of nc functions. A nc function defined on a free open set Ω is said to *nc holomorphic* if it is locally bounded. It is a remarkable fact that a nc holomorphic function is holomorphic in a natural analytic sense [48].

For every nc set Ω we let $H^\infty(\Omega)$ be the algebra of bounded nc functions on Ω equipped with the supremum norm $\|f\| = \sup_{X \in \Omega} \|f(X)\|$, and we let $A(\Omega)$ be the closure of the polynomials in $H^\infty(\Omega)$ with respect to the supremum norm. A *nc variety* is the zero set of H^∞ functions, that is a free set of the form $\mathfrak{V} = \{X \in \Omega : \forall f \in S \, . \, f(X) = 0\}$, where $S \subseteq H^\infty(\Omega)$.

In [65], Guy Salomon (my PhD student), Eli Shamovich (my postdoc) and I initiated the study of bounded nc functions on the nc unit ball and its subvarieties. We identified algebras of nc functions as multiplier algebras of nc reproducing kernel Hilbert spaces [14], proved they have the complete Pick property, and classified them. The classification program for the algebras $H^\infty(\mathfrak{V})$ is modeled on and extends the isomorphism problem of Section 2.1. The following result is representative.

Theorem 16 (Salomon, Shalit and Shamovich [65])**.** *Let $\mathfrak{V} \subseteq \mathfrak{B}_d$ and $\mathfrak{W} \subseteq \mathfrak{B}_e$ be nc varieties. $H^\infty(\mathfrak{V})$ and $H^\infty(\mathfrak{W})$ are completely isometrically isomorphic if and only if there exists a nc holomorphic map $G : \mathfrak{B}_e \to \mathfrak{B}_d$ and a nc holomorphic map $H : \mathfrak{B}_d \to \mathfrak{B}_e$ such that $G\big|_{\mathfrak{W}} = (H\big|_{\mathfrak{V}})^{-1}$.*

We also showed that a completely isometric isomorphism $\alpha : H^\infty(\mathfrak{V}) \to H^\infty(\mathfrak{W})$ is implemented by some G as $\alpha(f) = f \circ G$. For homogeneous varieties we showed that G and H can be chosen to be automorphisms in $\mathrm{Aut}(\mathfrak{B}_d) \cong \mathrm{Aut}(\mathbb{B}_d)$. Shamovich later extended this to almost full generality [75]. This is a noncommutative version of Theorem 14. For the free commutative ball $\mathfrak{C}\mathfrak{B}_d$ consisting of all commuting strict row contractions, $H^\infty(\mathfrak{C}\mathfrak{B}_d) = \mathcal{M}_d$, the multiplier algebra of H^2_d. If $\mathfrak{V}$ is a subvariety of $\mathfrak{C}\mathfrak{B}_d$ then $H^\infty(\mathfrak{V})$ is a quotient of $\mathcal{M}_d$. This quotient can be an algebra of the form $\mathcal{M}_V$ which corresponded to "radical" ideals, but it can also be an algebra that encodes multiplicity.

The paper [65] contained also similar results for the algebras $A(\mathfrak{V})$ defined to be the closure of the polynomials in $H^\infty(\mathfrak{V})$ with respect to the supremum norm $\|f\| = \sup_{X \in \mathfrak{V}} \|f(X)\|$, under the assumption that $\mathfrak{V}$ is homogeneous. Interestingly, we also show that $A(\mathfrak{V})$ is equal to the algebra of all uniformly continuous nc functions on $\mathfrak{V}$.

What about the noncommutative version of Theorem 15, that is, classification of $H^\infty(\mathfrak{V})$ up to isomorphism? In [66] Salomon, Shamovich and I approached this problem. Motivated by the commutative case, one might guess that the variety $\mathfrak{V}$ is an invariant for the algebra $H^\infty(\mathfrak{V})$, with the difference that the class of morphisms determining the geometry should be nc biholomorphisms rather than nc automorphisms. But it turns out that if there is a nc biholomorphism between two nc subvarieties $\mathfrak{V}, \mathfrak{W} \subseteq \mathfrak{B}_d$, then there is a nc automorphism $F \in \mathrm{Aut}(\mathfrak{B}_d)$ such that $F(\mathfrak{V}) = \mathfrak{W}$. But that would imply that the algebras are completely isometrically isomorphic. Thus, the nc variety $\mathfrak{V}$ cannot be a complete invariant of $H^\infty(\mathfrak{V})$, it is *not even an invariant* of the algebra!

We show that the similarity envelope $\widetilde{\mathfrak{V}} = \cup_n \widetilde{\mathfrak{V}}_n$ of $\mathfrak{V}$, where

$$\widetilde{\mathfrak{V}}_n = \{(S^{-1}X_1S, \ldots, S^{-1}X_dS) : X = (X_1, \ldots, X_d) \in \mathfrak{V}_n \text{ and } S \in GL_n\}.$$

parameterizes the space of finite dimensional weak-$*$ representations of $H^\infty(\mathfrak{V})$, by way of

$$\widetilde{\mathfrak{V}} \ni X \longleftrightarrow \Phi_X : f \mapsto f(X).$$

We view the similarity envelopes as metric spaces by introducing a metric δ on the similarity envelope of the ball $\widetilde{\mathfrak{B}_d}$ that we call the *nc pseudohyperbolic metric*, given by

$$\delta(X, Y) = \|\Phi_X - \Phi_Y\|.$$

A representative classification result is the following.

Theorem 17 (Salomon, Shamovich and Shalit [66])**.** *Let $\mathfrak{V} \subseteq \mathfrak{B}_d$ and $\mathfrak{W} \subseteq \mathcal{B}_e$ be two homogeneous nc varieties. The following statements are equivalent:*

(1) *$H^\infty(\mathfrak{V})$ and $H^\infty(\mathfrak{W})$ are weak-$*$ continuously isomorphic.*
(2) *$H^\infty(\mathfrak{V})$ and $H^\infty(\mathfrak{W})$ are boundedly isomorphic.*
(3) *$H^\infty(\mathfrak{V})$ and $H^\infty(\mathfrak{W})$ are completely boundedly isomorphic.*
(4) *There exists a bi-Lipschitz nc biholomorphism mapping $\widetilde{\mathfrak{W}}$ onto $\widetilde{\mathfrak{V}}$.*
(5) *There exists a bi-Lipschitz linear map mapping $\widetilde{\mathfrak{W}}$ onto $\widetilde{\mathfrak{V}}$.*

In addition, any isomorphism that appears in (1)–(3) can be viewed as a pre-composition with a bi-Lipschitz nc biholomorphism between the similarity envelopes.

In the general case of not-necessarily-homogeneous varieties, we have that (1) is equivalent to (4) and then (2) is automatic, as well as a version for completely bounded weak-$*$ isomorphisms.

Some of the tools that are developed to obtain Theorem 17 are interesting in their own right. For example, we proved the following nc counterpart of the Schwarz Lemma of the disc.

Theorem 18 (Salomon, Shamovich and Shalit [66])**.** *Let $f \colon \mathbb{D} \to \widetilde{\mathfrak{B}_d}$ be a holomorphic function mapping 0 to 0, and let ρ denote the joint spectral radius of a d-tuple of matrices. Then*

(1) *$\rho(f(z)) \leq |z|$ for every $z \in \mathbb{D}$ and $\rho(f'(0)) \leq 1$; and*
(2) *if $f'(0)$ is an irreducible coisometry, then $f(z)$ is similar to $zf'(0)$ for very $z \in \mathbb{D}$.*

We used the above theorem to prove a nc version of Cartan's uniqueness theorem for similarity envelopes. Recall that by Cartan's uniqueness theorem if $F \colon U \to U$ is holomorphic on a bounded domain in $\mathbb{C}^d$ such that $F(a) = a$ and $DF(a) = \mathbf{id}$ for some $a \in U$, then $F = \mathbf{id}_U$. This cannot hold for maps on $\widetilde{\mathfrak{B}_d}$ because every similarity in $H^\infty(\mathfrak{B}_d)$ is an automorphism, so by Theorem 17 it induces a map $G \colon \widetilde{\mathfrak{B}_d} \to \widetilde{\mathfrak{B}_d}$ that fixes first level. However, we have the following.

Theorem 19 (Salomon, Shamovich and Shalit [66])**.** *Let $G : \widetilde{\mathfrak{B}_d} \to \widetilde{\mathfrak{B}_d}$ a nc holomorphic map that fixes the origin and such that its derivative at 0 is the identity. If $X \in \widetilde{\mathfrak{B}_d}$ is irreducible, then $G(X)$ is similar to X.*

Our motivation for proving this theorem is the attempt to understand the automorphisms of $H^\infty(\mathfrak{B}_d)$; indeed, by Theorem 17 these are determined by biholomorphic automorphisms of $\widetilde{\mathfrak{B}_d}$. This also connects to a problem going back to Davidson and Pitts on the automorphisms of the *noncommutative analytic Toeplitz algebras* [27]. We showed that theses algebras studies by Davidson and Pitts are all isomorphic to some $H^\infty(\mathfrak{B}_d)$. Translated to our language, Davidson and Pitts proved Theorem 16 for the case where $\mathfrak{V} = \mathfrak{W} = \mathfrak{B}_d$, namely they proved that every $G \in \mathrm{Aut}(\mathbb{B}_d)$ gives rise to a completely isometric automorphism of $H^\infty(\mathfrak{B}_d)$, and that all completely isometric automorphisms arise this way. They showed that there is a surjection $\tau\colon \mathrm{Aut}(H^\infty(\mathfrak{B}_d)) \to \mathrm{Aut}(\mathbb{B}_d)$ with a continuous section $\kappa : \mathrm{Aut}(\mathbb{B}_d) \to \mathrm{Aut}(H^\infty(\mathfrak{B}_d))$, taking $\mathrm{Aut}(\mathbb{B}_d)$ onto the completely isometric automorphisms of $H^\infty(\mathfrak{B}_d)$.

Davidson and Pitts referred to the kernel of τ as *quasi-inner* automorphisms, these are precisely the automorphisms that fix the character space. In our language, these are the automorphisms whose implementing map G fixes the first level of the nc ball. They asked whether all quasi-inner automorphisms are inner, i.e. implemented by a conjugation with an invertible element in $H^\infty(\mathfrak{B}_d)$. We give a partial answer in Theorem 19, which says that every quasi-inner is "pointwise inner". If the similarity in Theorem 19 is implemented by an invertible element $\varphi(X)$ where $\varphi \in H^\infty(\mathfrak{B}_d)$ is independent of X, this would show that all quasi-inner automorphisms are inner.

2.4. **von Neumann's inequality for row contractive matrix tuples and applications to operator algebras and nc function theory.** During our investigation of continuous nc functions in [65] the question arose, whether there exists a uniformly bounded nc holomorphic function on the free commutative ball $\mathfrak{CB}_d$ that is levelwise uniformly continuous but not globally uniformly continuous. This has led me to the collaboration [40], joint with Michael Hartz and Stefan Richter. It turns out that the question on uniformly continuous nc functions on $\mathfrak{CB}_d$ can be reduced to the following von Neumann type inequality, which we prove.

Theorem 20 (Hartz, Richter and Shalit [40])**.** *There exists a constant C_n such that for all $d \in \mathbb{N}$, for every commuting row contraction $T = (T_1, \ldots, T_d)$ on a Hilbert space of dimension n and for every polynomial $p \in \mathbb{C}[z_1, \ldots, z_d]$, the inequality*

$$\|p(T)\| \leq C_n \sup_{z \in \mathbb{B}_d} |p(z)| \tag{10}$$

holds.

It is known that there is no constant that would work for all n. Inequality (10) might seem elementary on first sight[1], but its proof required novel ideas, and it has led to the resolution of several open questions. Besides the the resolution in the affirmative of the above mentioned problem on uniformly continuous nc functions, we applied our results to show that (i) Gleason's problem cannot be solved contractively in $H^\infty(\mathbb{B}_d)$ for $d \geq 2$, (ii) the multiplier algebra $\mathrm{Mult}(\mathcal{D}_a(\mathbb{B}_d))$ of the weighted Dirichlet space $\mathcal{D}_a(\mathbb{B}_d)$ on the ball is not *topologically subhomogeneous* when $d \geq 2$ and $a \in (0, d)$ — an open problem from [4]; and (iii) we determined the bounded finite dimensional representations of the norm closed subalgebra $A(\mathcal{D}_a(\mathbb{B}_d))$ of $\mathrm{Mult}(\mathcal{D}_a(\mathbb{B}_d))$ generated by polynomials. In particular, we determined the bounded representations of the *ball algebra* $A(\mathbb{B}_d)$, that is, the closure of the analytic polynomials in $C(\overline{\mathbb{B}}_d)$ with respect to the supremum norm.

Theorem 21 (Hartz, Richter and Shalit [40])**.** *For all $a > 0$, the unital bounded n-dimensional representations of $A(\mathcal{D}_a(\mathbb{B}_d))$ coincide with those of $A(\mathbb{B}_d)$, and these are precisely the maps $f \mapsto f(T)$ where T is a d-tuple jointly similar to a row contraction.*

2.5. **Classification and representation of operator algebras of subproduct systems.** In this section, by a *subproduct system* we mean a family $X = \{X_n\}_{n\in\mathbb{N}}$ of Hilbert spaces where $X_0 = \mathbb{C}$ and, roughly,

$$X_{m+n} \subseteq X_m \otimes X_n$$

for all $m, n \in \mathbb{N}$. Let $\mathcal{F}(X) = \bigoplus_{n=0}^\infty X_n$ be the *Fock space* over X. The *tensor algebra* $\mathcal{A}_X$ is the norm closed algebra generated by the operators S_ξ, $\xi \in X_m$, given by the shift operators

$$S_\xi(\eta) = P_{X_{m+n}}(\xi \otimes \eta) \quad , \quad \eta \in X_n$$

[1]Go ahead, make my day.

where $P_{X_{m+n}}$ denotes the projection of $X_m \otimes X_n$ onto X_{m+n}. These algebras are related to the *tensor algebras* $\mathcal{T}_E^+$ of Muhly-Solel [55], and the C*-algebras arising from subproduct systems are counterparts of the *Toeplitz-Pimsner algebras* $\mathcal{T}_E$ and *Cuntz-Pimsner algebras* $\mathcal{O}_E$ of Pimsner and Katsura [63, 49]. In [74, 28, 47] we worked on the problem of classifying the nonselfadjoint algebras $\mathcal{A}_X$ for interesting classes of subproduct systems. We also studied related C*-algebras and we identified the C*-envelope of $\mathcal{A}_X$.

The main classification result was, roughly, this: *For two subproduct systems X and Y with finite dimensional fibers, the tensor algebras $\mathcal{A}_X$ and $\mathcal{A}_Y$ are isometrically isomorphic if and only if X and Y are isomorphic subproduct systems* [28]. In [47] we proved a similar result about bounded isomorphism. The corresponding question about systems with infinite dimensional fibers was left open because the proof used inherently finite dimensional complex geometry.

In [65] we showed that if X is a subproduct system with $\dim X_1 = d \in \mathbb{N}$, then there exists a homogeneous ideal $J \lhd \mathbb{C}\langle z_1, \ldots, z_d\rangle$ and a homogeneous nc variety

$$\mathfrak{V} = V(J) := \{Z \in \mathfrak{B}_d : p(X) = 0 \text{ for all } p \in J\} \tag{11}$$

such that $\mathcal{A}_X = A(\mathfrak{V})$, in other words, in finitely many variables tensor algebras of subproduct systems are just algebras of nc functions. Thus we were led to two questions regarding subproduct systems with infinite-dimensional fibers:

(1) Is it true, that two tensor algebras are isomorphic (in an appropriate sense) if and only if their subproduct systems are isomorphic (in the corresponding sense)?
(2) Can every tensor algebra be identified with the algebra of uniformly continuous nc functions on a homogeneous nc variety?

Michael Hartz and I set to resolve these problems. There is a bijective correspondence between subproduct systems X and homogeneous ideals J in $A(\mathfrak{B}_d)$. With every homogeneous ideal J we may also define a variety $V(J)$ as in (11). The closure $\overline{V(J)}$ corresponds to all bounded finite dimensional representations of $\mathcal{A}_X$, therefore every element in $\mathcal{A}_X$ can be considered as a uniformly continuous nc function on $V(J)$. This gives a natural restriction map $\mathcal{A}_X \to A(V(J))$ and the question becomes whether this is a complete isomorphism.

Let us say that a closed idea $J \lhd A(\mathfrak{B}_d)$ *satisfies the Nullstellensatz* if

$$J = I(V(J)) := \{f \in A(\mathfrak{B}_d) : f(X) = 0 \text{ for all } X \in V(J)\}.$$

Theorem 22 (Hartz and Shalit [41])**.** *Let J be a homogeneous ideal in $A(\mathfrak{B}_d)$ and let X the corresponding subproduct system. The following are equivalent:*

(1) *The ideal J satisfies the Nullstellensatz.*
(2) *The restriction map $\mathcal{A}_X \to A(V(J))$ is injective.*
(3) *The restriction map $\mathcal{A}_X \to A(V(J))$ is a completely isometric isomorphism.*
(4) *$\mathcal{A}_X$ is residually finite dimensional.*

It is known that when $d < \infty$ every homogeneous ideal satisfies the Nullstellensatz $J = I(V(J))$ [65]. However, in [41] we found that for $d = \infty$ this might fail. It follows that the class of tensor algebras of subproduct systems is richer than the class of uniformly continuous nc functions on homogeneous varieties. In particular, one cannot reduce the isomorphism problem for tensor algebras to function theory. We solved the problem by a different route, finally extending my results with Davidson and Ramsey [28] and with Kakariadis [47] to the case $d = \infty$.

Theorem 23 (Hartz and Shalit [41])**.** *For subproduct systems X and Y the following are equivalent:*

(1) *There exists a bounded isomorphism $\varphi \colon \mathcal{A}_X \to \mathcal{A}_Y$.*
(2) *There exists a completely bounded isomorphism $\varphi \colon \mathcal{A}_X \to \mathcal{A}_Y$.*
(3) *There exists a similarity $W : X \to Y$.*

Also, the following are equivalent:

1. *There exists an isometric isomorphism $\varphi\colon \mathcal{A}_X \to \mathcal{A}_Y$.*
2. *There exists a completely isometric isomorphism $\varphi\colon \mathcal{A}_X \to \mathcal{A}_Y$.*
3. *There exists an isomorphism $W\colon X \to Y$.*

2.6. **My past work on the Arveson-Douglas essential normality conjecture.** The Arveson-Douglas essential normality conjecture has intrigued me for several years, but I did not obtain new results on this topic in the last six years. Still, it is perhaps worth noting that some of my results have survived. This more-than-twenty-year-old open problem has attracted the attention of many researchers, some of them outstanding (see the survey [37]). The problem is, roughly, whether all graded quotient modules of the Drury-Arveson Hilbert module H^2_d are essentially normal. This conjecture was resolved for several special cases. In the first decade of this century, progress was made mainly on the case of monomial ideals, principal ideals, and low or high dimensional cases. In the last decade, most progress has been made using hard harmonic analytical methods, the conjecture was shown to hold under various smoothness assumptions, and results were extended for Hilbert modules other than the Drury-Arveson space, as well as for Hilbert modules of holomorphic function on pseudo-convex domains beyond the unit ball. But progress was limited in the direction of reducible varieties. In [68] I introduced the *stable division property* (developed further with Biswas in [22]), which inspired Kennedy to approach the problem from a decomposability perspective [50] and this led us afterwards to collaborate and write [51], where, using mostly operator theoretic methods, we solved the problem for quotient modules corresponding to a variety that is the union of linear subspaces — this might still be the best result in this particular direction.

In [52] Kennedy and I approached the problem from a completely different, operator algebraic, approach and we showed that a quotient of H^2_d is essentially normal if and only the corresponding shift is hyperrigid in the sense of Arveson [11]. This paper is unique in the following way: rather than solving the conjecture for a special case, as most papers on the subject proceed, it establishes weaker forms (consequences of) the conjecture for *all* graded quotients; for example we identify the correct C*-envelope of the algebra generated by the quotient module. As far as I know, there has not been any *general* progress on the conjecture since then.

3. Dilation theory of completely positive semigroups

A *CP-semigroup* is a semigroup $\varphi = \{\varphi_t\}_{t\geq 0}$ of completely positive maps on a unital C*-algebra $\mathcal{B}$. When each φ_t is a $*$-endomorphism of $\mathcal{B}$, then φ is an *E-semigroup*. An E-semigroup α on an algebra $\mathcal{A}$ is said to be an *E-dilation* for φ if there is some projection $p \in \mathcal{A}$ such that $\mathcal{B} = p\mathcal{A}p$ and

$$\varphi_t(b) = p\alpha_t(b)p, \tag{12}$$

for all $b \in \mathcal{B}$ and $t \in \mathbb{R}_+$. A pivotal result in this field is Bhat's theorem, which says, roughly, that every (one-parameter) CP-semigroup has a unique E-dilation [18, 19, 56].

My main contribution to this field is the extension of Bhat's Theorem to CP-semigroups $\varphi = \{\varphi_s\}_{s\in\mathcal{S}}$ which are parameterized by some semigroup $\mathcal{S}$, rather then by the semigroup $\mathbb{R}_+$. The cases of greatest interest are $\mathcal{S} = \mathbb{N}^k$ or $\mathcal{S} = \mathbb{R}^k_+$. My first early results were that every pair of *strongly commuting* CP-semigroups has an E-dilation [67, 69]. Then, in order to study CP-semigroups over general semigroups, together with my PhD supervisor Baruch Solel, we introduced the notion of a *subproduct system*, a generalization of Arveson's *product systems* [9]. A *subproduct system* is a family $E = \{E_s\}_{s\in\mathcal{S}}$ of C*-correspondences that satisfy, roughly,

$$E_{s+t} \subseteq E_s \otimes E_t \ , \quad \text{for all} \quad s,t \in \mathcal{S}. \tag{13}$$

A *product system* is a subproduct system where in (13) equality holds instead of inclusion. We showed that there is a bijective correspondence between CP-semigroups of normal maps on a von Neumann algebra $\mathcal{B}$, on the one hand, and subproduct systems of W*-correspondences (over $\mathcal{B}'$) and subproduct system representations, on the other [74]. This correspondence allows to use subproduct systems as the main tool for either constructing dilations or for showing that they do not exist.

For CP-semigroups that act on a C*-algebra the machinery developed in [74] does not apply. For over a decade I worked with Michael Skeide on developing an alternative theory. This project was completed in 2020, and was accepted for publication last year [73]. This paper is over 200 pages long and very rich with results, examples and counter-examples. Here I'll only give some highlights, being necessarily imprecise. The following captures the central theme of the paper.

Theorem 24 (Shalit and Skeide [73])**.** *To every CP-semigroup $\varphi = \{\varphi_s\}_{s\in\mathcal{S}}$ on a C*-algebra $\mathcal{B}$, there corresponds a subproduct system of $\mathcal{B}$-correspondences $\{E_s\}_{s\in\mathcal{S}}$, and a unit $\{\xi_s \in E_s\}_{s\in\mathcal{S}}$ that satisfies $\xi_{s+t} = \xi_s \otimes \xi_t$ such that φ is represented as*

$$\varphi_s(b) = \langle \xi_s, b\xi_s \rangle \quad , \quad b \in \mathcal{B}, s \in \mathcal{S}. \tag{14}$$

If φ is unital, then it has a full and strict unital E-dilation, if and only if the subproduct system embeds into a product system.

The representation in (14) is just the well-known GNS representation of a CP map, and the observation that it assembles into a subproduct system is easy. The meat of the theorem is in the last line. Still, this theorem is not so far from the machinery of Bhat and Skeide [19] powered by the insight gained in [74], and the real difficulty starts when we wish to apply it effectively to various situations, and in particular when we wish to treat *all* dilations, and not only those that are *full and strict* as required in the theorem, or if we consider nonunital CP-semigroups. One of our novel results is the identification of the role that *superproduct systems* (where the inclusion in equation (13) is reversed) play in dilation theory.

Theorem 25 (Shalit and Skeide [73])**.** *A CP-semigroup has a good dilation only if its subproduct system embeds into a superproduct system.*

We use this theorem to prove that there are three commuting unital CP maps with no dilation *whatsoever*. This improves on previous negative results in this direction [72, 74], that could only rule out the existence of full and strict dilations.

Theorem 26 (Shalit and Skeide [73])**.** *There exist three commuting unital CP maps for which there is no dilation whatsoever.*

It is interesting to note that although we provide an explicit, concrete and finite dimensional counter example, the proof involves passing from the GNS subproduct system via Morita equivalence to another subproduct system that we associate with every CP-semigroup. We also have new results in the positive direction, concerning the case $\mathbb{N}^2$ (two commuting maps) and $\mathbb{R}_+^k$ (quantized convolution semigroups).

The most difficult part of the paper is perhaps the one dealing with minimality (Sections 21 and 22). There are several natural notions of *minimality* for dilations. We study them and compare them using our machinery. We use our framework to analyze the dilation that we constructed in the case of two commuting normal UCP maps, and we show that, in general, the natural notions of minimality differ in the multi-parameter case. Worse: minimal dilations are not unique. In fact, we show that there are dilations that cannot be *minimalized* in a certain sense, contrary to what anyone might guess. We also analyze the one-parameter case and we obtain the following result.

Theorem 27 (Shalit and Skeide [73])**.** *For a normal strong dilation of a one-parameter CP-semigroup, the following are equivalent:*

(1) *The dilation is fully minimal.*
(2) *The dilation is incompressible.*
(3) *The dilation is strongly incompressible.*

Every dilation can be restricted and compressed to the minimal dilation, which is unique.

This improves in two ways on a result of Arveson, who obtained the equivalence of (1) and (3) in the unital case [10, Section 8.9]. It is very pleasing that our framework, which was introduced in order to treat the difficulties that arise in the multi-parameter case, also found applications in new results for the one-parameter case.

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