E

הקרן הלאומית למדע
المؤسّسة الإسرائيلية للعلوم
Israel Science Foundation

Personal Research Grants
Research Grant Application no. 431/20

General application information

Role	Name	Academic Rank	Department	Institute
PI.1	Orr Moshe Shalit	Associate Professor	Mathematics	Technion

Research Title

Noncommutative analysis, dilation theory and operator algebras

Keywords

nc function theory, Operator Algebras, matrix convexity, Completely Positive Maps, operator systems, dilation

Requested Budget in NIS

4	283,549 NIS
No. of Years	Average Annual Budget

Authorization of the Institution

Name & Position	Signature & Stamp	Date

Application No.: 431/20
PI name: Orr Moshe Shalit

Scientific abstract: **Noncommutative analysis, dilation theory and operator algebras**

I propose to study several concrete problems within the broad context of noncommutative analysis. The key objects of study will be noncommutative (nc) spaces and functions, and the focus will be put on two kinds of models of nc function theory: the first revolving around matrix convexity, and the second algebras of nc holomorphic functions on nc varieties. These two objectives can be considered as the “real" and “complex" (analytic) versions of noncommutative analysis, respectively.

Noncommutative real analysis. I will outline a plan to study the connections between operator spaces and the geometry of matrix convex sets and matrix ranges, with emphasis on two novel ingredients.

First, dilations will play a central and modified role. We will study how dilations and the Choquet and Shilov boundaries of operator spaces reflect in the extreme geometry of matrix convex sets, and search for concrete geometric characterization of operator-theoretic properties, such as hyperrigidity. Dilations have further applications, revealing more structure than previously believed. For *any* pair of d-tuples $A \in B(\mathcal{H})^d$ and $B \in B(\mathcal{K})^d$, we seek the optimal constant $c(A, B)$, such that a $*$-isomorphic copy of $c(A, B) \cdot B$ is a dilation of A. The constants $c(A, B)$ can serve as a measure of distance between the operator algebras generated by the tuples, and this point of view has already found surprising uses, e.g. to the continuity of spectra of Schrodinger operators. This avenue will be developed.

A second novel ingredient that we propose, is the incorporation of free probability and random matrix theory to the study of nc convex sets. Such a perspective allows us to ask and answer exciting new questions, such as: *what does a typical matrix range tend to look like?*

Noncommutative complex analysis. A major objective of my previous proposal was to study algebras of bounded nc analytic varieties in the nc unit ball, and to classify them up to various kinds of isomorphisms, in nc geometric terms. This goal was achieved to a large extent, and one of the successful byproducts was the development of nc function-theoretic tools, mainly in the context of the nc unit ball. In the current proposal, I am setting the goal of pushing all of these ambitions and results beyond the nc unit ball. Explicitly, I will study algebras of bounded nc analytic functions on general nc domains and varieties. On the operator algebraic side, we wish to obtain a classification scheme that covers a far richer class of nonselfadjoint operator algebras than has been treated before. It is my hope — which is grounded on experience — that this pursuit will also serve as an engine and a guiding light for the development of the theory of analytic nc functions in several complex variables beyond its current frontiers.

Research program

1. Scientific Background

1.1. **Overview.** *Noncommutative analysis* is concerned with the study of "noncommutative" objects (e.g. operator algebras) by making use of ideas that originate in the study of "commutative" objects (e.g. function algebras on topological spaces). On the one hand, a major goal is the generalization of notions from the commutative to the noncommutative world, providing tools by which to pose and tackle questions on noncommutative objects. On the other hand, one hopes that placing a commutative entity within the extensive noncommutative perspective, will illuminate classical problems.

Within this broad perspective, I plan to examine several concrete problems in noncommutative (nc) analysis. The key objects of study will be nc spaces and functions, and the focus will be put on two models: the first emphasizing nc convexity, and the second algebras of nc analytic functions on nc varieties. These can be considered as the "real" and "complex" versions of noncommutative analysis.

For both the real and the complex versions of nc analysis, there are several different competing frameworks and points of view. A special role is played by the the abstract, "all encompassing" theories. In Gelfand's spirit, every algebra can be thought of as an algebra of functions on its representation space. In the 1940s, Gelfand and Naimark proved that every commutative C*-algebra is an algebra of continuous functions on a topological space. In the 1960s, Takesaki and then Bichteler proved that every C*-algebra is an algebra of continuous "functions" on its space of representations. Davidson and Kennedy recently developed an extensive theory of nc functions [31], using Takesaki and Bichteler's ideas to identify the continuous nc functions on a nc convex set K as the maximal C*-algebra generated by the continuous nc affine functions on K. They generalize much of Choquet theory to the nc setting.

On the analytic side, Taylor developed a very general theory of nc functions [104, 105]. Taylor's noncommutative function theory grew out of his analytic functional calculus for tuples of commuting operators. His starting point was the free algebra generated by d noncommuting elements. Certain topological closures of this free algebra, called *localizations*, are the algebras of nc functions that one would apply to tuples. Another attempt to develop nc function theory is due to Voiculescu.

With the abstract theories in mind, and perhaps also at hand, I will work in somewhat more specialized frameworks, that have been developed in the past two decades. For analytic nc function theory I will follow Kalyuzhnyi-Verbovetskii and Vinnikov [62], for whom a nc analytic function in d variables is a graded map from d-tuples of $n \times n$ matrices in some "nc domain" that respects similarity. There is overlap between this approach and other contributions made by Agler-McCarthy [2, 3, 4], Davidson-Pitts [32, 33], Helton-Klep-McCullough [49], Muhly-Solel [75, 76, 78, 79], Popescu [90, 91, 92, 93, 94, 95, 96], Jury-Martin [59, 60] and others (e.g. [15, 80, 81, 98, 99, 103]). The works of Ball-Marx-Vinnikov on nc reproducing kernel Hilbert spaces [17, 18], and of Agler-McCarthy-Young on nc manifolds [5] are important recent developments.

For nc convexity, the framework we will use is by and large the one introduced by Wittstock [108] and revived by Effros-Winkler [38], which has served several groups of researchers, e.g. Helton-Klep-McCulough and co. [39, 40, 41, 51, 53, 54, 109] as well as others [42, 43, 66, 82, 83, 84, 107, 109].

My proposal consists of a series of problems that fall under the unifying theme of noncommutative analysis. Some problems can be classified as "nc convexity" and some as "nc (analytic) function theory". Each one is of inherent interest as well as possible applicability; the program as a whole connects to the work of various noncommutative tribes, as well as to open problems. I believe that the organic unity of this proposal will unfold as the problems are described below.

1.2. **Basic definitions and notation.**

1.2.1. *NC sets.* Let $M_n = M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices over $\mathbb{C}$, and let M_n^d be the set of all d-tuples of such matrices. The "noncommutative universe" in which most of the action takes place in is the disjoint union $\mathbb{M}^d = \cup_{n=1}^\infty M_n^d$. A subset $\Omega \subseteq \mathbb{M}^d$ is called a *free set.* A free set Ω is said to be a *nc set* if it is closed under direct sums and conjugation with unitaries. If Ω is a nc set, we denote $\Omega_n = \Omega \cap M_n^d$, and $\Omega = \cup_{n=1}^\infty \Omega_n$. The *similarity envelope* of a nc set Ω is the nc set $\widetilde{\Omega}$ consisting of all tuples $S^{-1}XS$ similar to some $X \in \Omega$ (here and below $S^{-1}(X_1, \ldots, X_d)S = (S^{-1}X_1S, \ldots, S^{-1}X_dS)$).

1.2.2. *NC holomorphic functions.* A function f from a nc set $\Omega \subseteq \mathbb{M}^d$ to $\mathbb{M}^1$ is said to be a *nc function* if: (i) f is graded: $X \in \Omega_n \Rightarrow f(X) \in M_n$; (ii) f respects direct sums: $f(X \oplus Y) = f(X) \oplus f(Y)$; and (iii) f respects similarities: if $X \in \Omega_n$, $S \in GL_n$, and if $S^{-1}XS \in \Omega_n$, then $f(S^{-1}XS) = S^{-1}f(X)S$.

A *free polynomial* is an element in $\mathbb{C}\langle z_1, \ldots, z_d\rangle$ — the free algebra in d variables. Free polynomials are the most important example of nc functions. To be specific, let $\mathbb{F}_d^+$ be the free semigroup generated by d generators $g_1, \ldots, g_d$. For a word $w = g_{i_1} \cdots g_{i_n} \in \mathbb{F}_d^+$, we write $z^w = z_{i_1} \cdots z_{i_n}$. A free polynomial is a finite sum $p(z) = \sum a_w z^w$, where $a_w \in \mathbb{C}$. Sometimes we just say *polynomial*, omitting "free". For $X = (X_1, \ldots, X_d) \in \mathbb{M}^d$ (or in fact for every $X \in B(\mathcal{H})^d$) we can evaluate p at X by plugging in: $p(X) = \sum a_w X^w$, where $X^w = X_{i_1} \cdots X_{i_n}$. A *matrix valued free polynomial* is a finite sum of the form $p(z) = \sum_w A_w z^w$, where $A_w \in M_n$. Evaluation at a tuple X is defined like so: $p(X) = \sum_w A_w \otimes X^w$.

A free set Ω is said to be open in the *disjoint union* topology if Ω_n is open in M_n^d for all n. The *free* topology is generated by *basic free open sets*, which are sets of the form $G_p := \{X \in \mathbb{M}^d : \|p(X)\| < 1\}$, where p is a matrix of polynomials. E.g., the *d-dimensional open matrix unit ball* $\mathfrak{B}_d$ is defined to be G_p for $p(z) = [z_1 \ z_2 \ \ldots \ z_d]$, thus $\mathfrak{B}_d = \{X \in \mathbb{M}^d : \|\sum X_j X_j^*\| < 1\}$. The *uniform* topology, defined naturally, is also useful. The choice of topology affects the following definition. A nc function defined on a free open set Ω is said to *nc analytic* (or *holomorphic*) if it is locally bounded. It turns out that a nc analytic function is an analytic function when considered as a function $f : \Omega_n \to M_n$, for all n, and moreover it has a nc "Taylor series" at every point [62]. A function $f : \Omega_n \to M_n$ is nc analytic w.r.t. the free topology if and only if it is locally approximable by polynomials [2].

1.2.3. *NC varieties.* A *nc algebraic variety* is a free set of the form $V_\Omega(S) = \{X \in \Omega : \forall p \in S \,.\, p(X) = 0\}$, where $S \subseteq \mathbb{C}\langle z_1, \ldots, z_d\rangle$. Likewise, let us define a *nc analytic variety in* Ω to be the joint zero set of a set of nc analytic functions on Ω. NC algebraic and analytic varieties are nc sets. We define $H^\infty(\Omega)$ to be the algebra of bounded holomorphic functions on Ω, and $A(\Omega)$ to be the algebra of bounded analytic functions that extend continuously to $\partial\Omega$ (and likewise for varieties).

1.2.4. *Matrix convex sets.* A nc set $\mathcal{S}$ is said to be *matrix convex* if it is closed under the application of UCP maps, i.e., whenever X is in $\mathcal{S}_n$ and $\phi \in \mathrm{UCP}(M_n, M_k)$, the tuple $\phi(X) := (\phi(X_1), \dots, \phi(X_d))$ is in $\mathcal{S}_k$. Equivalently, $\mathcal{S}$ is closed under *matrix convex combinations.* The main examples of matrix convex sets are *free spectrahedra* and *matrix ranges.* A *monic linear pencil* is a matrix valued free polynomial $L(x) = L_A(x) = I - \sum A_j x_j$, where $A \in B(\mathcal{H})^d$. We write $\mathcal{D}_L$ (or $\mathcal{D}_A$) for the union $\cup_n \mathcal{D}_L(n)$, where $\mathcal{D}_L(n) = \{X = (X_j) \in M_n^d : \mathrm{Re}\, L(X) \geq 0\}$. The set $\mathcal{D}_L$ is called a *free spectrahedron* (some authors reserve "free spectrahedron" for matrix coefficients). The *matrix range* [10] of a tuple A in $B(\mathcal{H})^d$ is the set $\mathcal{W}(A) = \cup_n \mathcal{W}_n(A)$, where $\mathcal{W}_n(A) = \{(\phi(A_1), \dots, \phi(A_d)) : \phi \in \mathrm{UCP}(C^*(S_A), M_n)\}$. A set $\mathcal{S} = \cup_n \mathcal{S}_n \subseteq \mathbb{M}^d$ is a bounded closed matrix convex set if and only if $\mathcal{S} = \mathcal{W}(A)$ for some A [28].

1.2.5. *Dilations and normal tuples.* A d-tuple $A \in B(\mathcal{H})^d$ is called a *compression* of $B \in B(\mathcal{K})^d$, and B is called a *dilation* of A, if there is an isometry $V : \mathcal{H} \to \mathcal{K}$ such that $A_i = V^* B_i V$ for $1 \leq i \leq d$; we then write $A \prec B$. Dilations are especially useful when the dilating tuple is normal. A tuple $N = (N_1, \dots, N_d)$ is called a *normal* tuple if $N_1, \dots, N_d$ are commuting normal operators. In this case the C^*-algebra generated by N is commutative, and is isomorphic to the algebra $C(X)$ of continuous functions on the joint spectrum $X = \sigma(N)$. If N acts on a finite dimensional space, $\sigma(N)$ is just the finite set of all d-tuples of joint eigenvalues.

1.3. **Background results in nc convexity.**

1.3.1. *Interpolation and inclusions of matrix convex sets.* The "nc universe" $\mathbb{M}^d$ is useful for studying problems that can be considered as "nc real algebraic geometry" such as the representation of positive nc functions as sums of squares [49, 50, 51, 53, 54, 55]. Such problems were shown by Helton-Klep-McCullough to be related to interpolation problems of unital completely positive (UCP) maps [51]. Interpolation problems for UCP maps, in turn, have been of great interest, because of their relationship to quantum information theory and theoretical physics, as well as with operator algebras [6, 24, 70]. Moreover, sums of squares problems are related to some very deep problems, from Hilbert's 17th problem ("classical" real algebraic geometry), to Connes' embedding problem [65]. Given two d-tuples of operators $A \in B(\mathcal{H})^d$ and $B \in B(\mathcal{K})^d$, the interpolation problem is: *when does there exist a UCP map $\phi : B(\mathcal{H}) \to B(\mathcal{K})$ such that $\phi(A_i) = B_i$ for all i?* For matrix tuples A, B, assuming that $\mathcal{D}_A$ is bounded, such a map exists if and only if $\mathcal{D}_A \subseteq \mathcal{D}_B$ [51]. Following Arveson [10], we showed that there exists a UCP map $\phi : A \to B$ if and only if $\mathcal{W}(B) \subseteq \mathcal{W}(A)$ [28]. We also observed that $\mathcal{W}(A)$ and $\mathcal{D}_A$ are related by the Effros-Winkler *matricial polar duality* [38] via $\mathcal{W}(A)^\circ = \mathcal{D}_A$, and from this we were able to deduce the above mentioned criterion for interpolation, as well as others, e.g., [70].

1.3.2. *Minimal and maximal matrix convex sets.* The results above motivate studying the problem of when one matrix convex set contains another. Given a closed convex set $K \subseteq \mathbb{C}^d$, the minimal matrix convex set $\mathcal{W}^{\min}(K)$ and the maximal matrix convex set $\mathcal{W}^{\max}(K)$ for which $\mathcal{W}_1^{\min}(K) = \mathcal{W}_1^{\max}(K) = K$ were described in several places; we follow [28] (see also [43], [52] and [87]; analogous constructions in the theory of operator spaces have appeared way back [85, 86]). One description for $\mathcal{W}^{\max}(K)$ is $\{X \in \mathbb{M}^d : \mathcal{W}_1(X) \subseteq K\}$. The minimal matrix convex set $\mathcal{W}^{\min}(K)$ (called the *matrix convex hull of* K in [52]) clearly exists, but here is a more useful description in terms of normal dilations (see [28]): $\mathcal{W}^{\min}(K) = \{X \in \mathbb{M}^d : \exists N \text{ normal, s.t. } X \prec N \text{ and } \sigma(N) \subseteq K\}$.

1.3.3. *Dilation constants.* In [84] we introduced the dilation constants $\theta(K,L) = \inf\{C > 0 : \mathcal{W}^{\max}(K) \subseteq C \cdot \mathcal{W}^{\min}(L)\}$ (where K, L are convex sets) and $\theta(K) := \theta(K,K)$. We computed $\theta(K,L)$ for various pairs. In particular, if $\overline{\mathbb{B}}_{p,d}$ is the closed unit $\|\cdot\|_p$-ball in $\mathbb{R}^d$, then $\theta(\overline{\mathbb{B}}_{p,d}) = d^{1-|1/p-1/2|}$. Moreover, we proved that $\theta(K) = 1$ if and only if K is a simplex, improving on a result of Fritz-Netzer-Thom who obtained this result for the case where K is a polytope. In fact, if $\mathcal{W}_n^{\max}(K) = \mathcal{W}_n^{\min}(K)$ for any $n \geq 2^{d-1}$ then K is a simplex. This was improved to $n \geq 2$ [57] in the case where K is a polytope.

1.3.4. *Dilations, boundaries, and matrix extreme points.* Matrix convex sets correspond to operator systems, and the matrix range of a tuple A determines the operator system it generates [10, 28, 43, 58, 84, 87]. In matrix convexity there are several notions of "extreme points". Webster and Winkler introduced the notion of *matrix extreme* point [107], and proved that a matrix convex set is the matrix convex hull of its matrix extreme points. Farenick proved that this notion corresponds to pureness of matrix states [42]. This played a role in Davidson and Kennedy's proof [30] of existence of *boundary representations* (aka *nc Choquet boundary* points [9, 10, 13]); these representations correspond to irreducible matrix extreme points that cannot be dilated non-trivially (this observation goes back to [37] and [74]), hence their proof relied on careful analysis of dilations. However, boundary representations might appear "at infinity", thus remain undetected in matrix convex sets of the type we consider. Evert-Helton-Klep-McCullough [41] defined a notion of *Arveson extreme point*, proved a Krein-Milman type theorem for these, and also a sort of "Milman's converse" [40] (which does not hold in general [39]). In [83], we proved that *incompressible* compact tuples are determined by their matrix ranges up to unitary equivalence, making use of all the above notions, as well as the C*-envelope [9, 10, 30, 37, 86].

1.4. **Background results in analytic nc function theory.**

1.4.1. *Universal algebras for relations.* In [102] we considered universal tuples for homogeneous ideals in noncommuting variables, and this led in [34, 46] to the classification up to isometric isomorphism of universal operator algebras generated by a commuting row contraction satisfying the relations in a homogeneous ideal. For a homogeneous ideal $I \triangleleft \mathbb{C}\langle z_1, \ldots, z_d\rangle$, denote by $\mathcal{A}_I$ the universal operator algebra generated by a row contraction S such that $p(S) = 0$ for all $p \in I$. We proved: $\mathcal{A}_I$ and $\mathcal{A}_J$ are isometrically isomorphic if and only if I and J are related by a unitary change of variables. The same results hold when one specializes to commuting variables, but then we can say more. For an ideal $I \triangleleft \mathbb{C}[z_1, \ldots, z_d]$, let $V_{\mathbb{B}_d}(I) = \{z \in \mathbb{B}^d : \forall p \in I. p(z) = 0\}$. If I and J are *radical* homogeneous ideals then $\mathcal{A}_I$ is isometrically isomorphic to $\mathcal{A}_J$ if and only if there is unitary mapping $V_{\mathbb{B}_d}(I)$ onto $V_{\mathbb{B}_d}(J)$; moreover, $\mathcal{A}_I$ and $\mathcal{A}_J$ are isomorphic if and only if there is a linear bijection $A : V_{\mathbb{B}_d}(J) \to V_{\mathbb{B}_d}(I)$. The same results hold for the WOT closures of the algebras. (See [7, 8, 61, 73, 92, 93] for related works in the noncommutative setting.)

1.4.2. *The isomorphism problem for complete Pick algebras.* The above problem in the commutative homogeneous radical case can be generalized significantly. Let H_d^2 be the Drury-Arveson space [11, 100] and $\mathcal{M}_d$ its multiplier algebra. Let $V \subseteq \mathbb{B}_d$ be an analytic variety determined by multipliers, and consider $\mathcal{M}_V = \mathcal{M}_d\big|_V$ (every complete Pick algebra arises this way [1]). In [29, 34, 35, 46, 47, 64, 72, 97] the isomorphism problem for the algebras $\mathcal{M}_V$ was investigated. It was shown that for every isometric isomorphism $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ there is a conformal automorphism $F \in \mathrm{Aut}(\mathbb{B}_d)$ such that $F(W) = V$, implementing φ as a composition operator $\varphi(f) = f \circ F$. Conversely, every $F \in \mathrm{Aut}(\mathbb{B}_d)$ gives rise to an isometric isomorphism. The algberaic isomorphism problem is more subtle: for every isomorphism $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ there exists a multiplier biholmorphism $F : W \to V$ which is bi-Lipschitz w.r.t. to the pseudohyperbolic metric, s.t. $\varphi(f) = f \circ F$. If V, W are homogeneous then F can be chosen linear. Conversely, under some assumptions, a biholmorphism $F : W \to V$ implements via composition an isomorphism $\mathcal{M}_V \to \mathcal{M}_W$. For example, if V, W are homogeneous, then the existence of a linear map sending W onto V bijectively implies the existence of an isomorphism [34, 46].

1.4.3. *Algebras of bounded nc analytic functions on subvarieties of the nc unit ball.* In [98, 99] we studied the extension of the above problem to the fully noncommutative case. In [98] we proved, roughly, that for two nc varities $\mathfrak{V}, \mathfrak{W} \subseteq \mathfrak{B}_d$, the algebras $H^\infty(\mathfrak{V})$ and $H^\infty(\mathfrak{W})$ are completely isometrically isomorphic if and only if there is a conformal automorphism $F \in \mathrm{Aut}(\mathfrak{B}_d) = \mathrm{Aut}(\mathbb{B}_d)$ from $\mathfrak{W}$ onto $\mathfrak{V}$ implementing the isomorphism as a composition operator (this also uses [103]). An interesting ingredient was a nc maximal modulus principle. In [99] we showed that $H^\infty(\mathfrak{V})$ and $H^\infty(\mathfrak{W})$ are weak-$*$ isomorphic, if and only if there is a nc analytic map $F : \widetilde{\mathfrak{W}} \to \widetilde{\mathfrak{V}}$ between the similarity envelopes of the varieties, that is bi-Lipschitz w.r.t. to the *nc pseudohyperbolic metric*. In the case of homogeneous varieties, the map can be chosen to be linear, and the weak-$*$ assumption can be dropped. A key ingredient there was a nc version of Cartan's uniqueness theorem for similarity envelopes.

2. Research objectives and expected significance

2.1. **Objectives.** My goal is *to study matrix convex sets and nc varieties through a series of concrete problems, and to investigate, via nc function theory and dilation theory, the connection between operator structures, on the one hand, and the geometry of nc convex sets and varieties, on the other*. Specifically:

(1) To construct normal dilations with prescribed spectra. In particular: to compute the dilation constant of a tuple of contractions to a tuple of commuting normals; to prove that a circumscribing simplex is always a minimal dilation hull.
(2) To develop and use dilation theory in fully noncommutative situations. To calculate sharp dilation constants $c(A, B)$ between interesting tuples A, B and link this to continuous fields.
(3) To connect the theory of nc convexity to random matrices and free probability theory. To discover "central limit theorems" for random matrix ranges. To obtain probablistic versions of von Neumann type inequalities and explore distributions of dilation constants.
(4) To connect geometric properties of the matrix range $\mathcal{W}(A)$ to operator-theoretic and operator space-theoretic properties of a d-tuple $A \in B(\mathcal{H})^d$. To detect *rigidity* or *hyperrigidity* of the operator system generated by A in terms of the nc geometry of the matrix range $\mathcal{W}(A)$. To solve Arveson's essential normality conjecture via hyperrigidity with these tools.
(5) To identify algebras of bounded nc analytic functions on general nc domains and varieties as operator algebras and study their structure and representation theory; to classify the algebras up to isomorphism by the geometric properties of the nc varieties/domains on which they live.
(6) To develop nc function theory in general nc domains, with emphasis on classification of domains/varieties, determining automorphism groups, and extension/approximation problems.

2.2. **Significance.** Studying the algebraic structure of the algebras of bounded free analytic functions will shed light on new and previously investigated isomorphism problems for universal operator algebras, and will stimulate developments in nc function theory. We believe that studying the structure of operator systems using geometry of matrix ranges may have the broadest impact beyond multivariable operator theory, touching upon semidefinite programming, quantum information theory, as well as operator algebras. Setting up a connection between nc convexity and random matrix theory/free probability may open up enitrely new avenues of research.

3. Detailed description of the proposed research

3.1. **Normal dilations and minimal dilation hulls.** We start with the following problem: *what is the smallest constant c, such that for every d-tuple of contractions A, there exists a normal tuple N of contractions such that $A \prec cN$?* If A is selfadjoint, then $c = \sqrt{d}$ works [84] (this found a surprising application in joint measurability of quantum effects [22]). If we let C_d denote the optimal constant, then it is known that $\sqrt{d} \leq C_d \leq \sqrt{2d}$ [82]. For $d = 2$ we have $C_2 \geq 1.54$ [44]. The value of C_d determines the best possible scale c such that the following von Neumann type inequality holds:

$$\|p(A)\| \leq \sup\{\|p(z)\| : z \in c\overline{\mathbb{D}}^d\} \tag{1}$$

for every tuple of contractions A and every matrix valued polynomial of degree ≤ 1. The precise value of C_d is a fundamental property of operator systems in general, and of the operator system spanned by the generators of the full C*-algebra of the free group in particular.

Still on the subject of dilations, let us define for every compact and convex $K \subseteq \mathbb{C}^d$,

$$\mathcal{E}(K) = \{L \supseteq K \text{ compact and convex } : \mathcal{W}^{\max}(K) \subseteq \mathcal{W}^{\min}(L)\}.$$

A minimal element of $\mathcal{E}(K)$ is called a *minimal dilation hull* of K. Also, let $\mathrm{md}(K) := \{L \in \mathcal{E}(K) : L \text{ is minimal}\}$. *How is a minimal dilation hull of K related to K?* It turns out that elements in $\mathrm{md}(K)$ need not resemble K in any way; there is no ball in $\mathrm{md}(\overline{\mathbb{B}}_{2,d})$, while $d \cdot \overline{\mathbb{B}}_{1,d} \in \mathrm{md}(\overline{\mathbb{B}}_{2,d})$. Also, there is a simplex in $\mathrm{md}(\overline{\mathbb{B}}_{2,d})$, so the shape of a minimal dilation hull is not unique [84]. I will work towards understanding dilation hulls, and, in particular, I plan to resolve the conjecture that *for every K, every circumbscribing simplex of K is a minimal dilation hull* (partial results in this direction were obtained in [82, 84]). This exciting connection between convex geometry and operator theory has several suggestive generalizations, on both sides of the containment $\mathcal{W}^{\max}(K) \subseteq \mathcal{W}^{\min}(L)$. For example, the concept of minimal dilation hull can be extended to finding, given a tuple $A \in B(\mathcal{H})^d$, minimal compact convex sets L such that $\mathcal{W}(A) \subseteq \mathcal{W}^{\min}(L)$; the geometry of L and its relation to $\mathcal{W}_1(A)$ carry information about the operator system generated by A. A problem with a similar flavour is: given a matrix convex set $\mathcal{S}$, one can study the maximal and minimal matrix convex sets $\mathcal{V}, \mathcal{U}$ such that $\mathcal{V}_n = \mathcal{S}_n = \mathcal{U}_n$ for some n. Similar problems have been considered for a while [52, 58, 87] and their solutions have applications to entanglement in quantum information theory.

3.2. **Dilations constants.** Given two d-tuples A and B, we extend the notataion $A \prec B$ to mean that A is a compression of a $*$-isomorphic copy of B; equivalently, there exist a UCP map such that

$B_i \mapsto A_i$ for all $i = 1, \ldots, d$. We let $c(A,B)$ be the minimal constant c such that $A \prec cB$. Note that

$$C_d = \max\{c(A,Z) : A \text{ is a tuple of contractions}\} = \max\{c(U,Z) : U \text{ is a tuple of unitaries}\},$$

where $Z = (Z_1, \ldots, Z_d)$ is the d-tuple of coordinate functions in $C(\overline{\mathbb{D}}^d)$. In our quest to find C_d, we considered the problem of finding the constants $c_\theta = c((u_\theta, v_\theta), (u_0, v_0))$, where (u_θ, v_θ) is the universal pair of unitaries that satisfy $v_\theta u_\theta = e^{i\theta} u_\theta v_\theta$, i.e., the generators of the rotations algebras [23]. In [44] we computed $c_\theta = \frac{4}{\|u_\theta + u_\theta^* + v_\theta + v_\theta^*\|}$, giving the lower bound $C_2 \geq \max_\theta c_\theta \approx 1.54 > \sqrt{2}$. It is interesting to note that the operator $h_\theta = u_\theta + u_\theta^* + v_\theta + v_\theta^*$ is a so-called *almost Mathieu operator* which is the subject of intensive research in mathematical physics (it is the Schrödinger operator appearing in the renowned context of Hosftadter's Butterfly [56]). We also computed constants $c_{\theta,\theta'} = c((u_\theta, v_\theta), (u_{\theta'}, v_{\theta'}))$, and found that $c_{\theta,\theta'} \leq e^{\frac{1}{4}|\theta - \theta'|}$. This has surprising applications: new proofs of the Lipschitz continuity of the norm $\theta \mapsto \|h_\theta\|$ [20] and the 1/2-Hölder continuity of the spectrum $\theta \mapsto \sigma(h_\theta)$ [16].

I propose to study dilation constants $c(U,V)$ for tuples of unitaries generating C*-algebras that carry a $\mathbb{T}^d$-action; we call these *generalized noncommutative tori.* They include rotation algebras, higher dimensional noncommutative tori, and also the reduced and full free group C*-algebras. These are fascinating constants in themselves, providing information about the elusive dilation constants C_d. I will exploit the hyperrigidity of operator systems generated by unitaries [14] to study $c(U,V)$ as a kind of metric between the different C*-algebras, study topological properties, and apply it to understand the relation between continuous fields and dilation theory.

3.3. **Probabilistic perspective.** To find the value of C_d we need to understand $c(U,Z)$, where Z is a normal tuple with sepctrum in $\overline{\mathbb{D}}^d$ and U is a unitary tuple. Numerical experimentation that we carried out with students [101] shows that if $U^{(n)} = (U_1^{(n)}, U_2^{(n)})$ is a "randomly" chosen pair of $n \times n$ unitaries, then $c(U^{(n)}, Z)$ tends to $\sqrt{2}$ as $n \to \infty$. We conjecture that for d-tuples of $n \times n$ Haar unitaries, $c(U^{(n)}, Z) \xrightarrow{n\to\infty} c(u_f, Z)$, where $u_f = (u_{f1}, \ldots, u_{fd})$ is the tuple of unitaries generating reduced C*-algebra of the free group, in line with [27]. One consequence is that, typically, d-tuples of contractions will satisfy the von Neumann type inequality (1) with a constant $c < C_d$.

The above observation opens the door for a completely fresh outlook on von Neumann type inequalities and dilation theory; in the classical theory [21] one asks about *worst case scenarios*, e.g., **every pair** of commuting contractions satisfies Ando's inequality, but **not every triplet** of commuting contractions satisfies a von Neumann type inequality. But we can ask *what is the probability that a von Neumann type inequality is satisfied?* This kind of question requires a probability distribution in order to make sense, and in the noncommutative setting there are natural candidates. A natural way to choose a "random" tuple of $n \times n$ unitaries $(U_1^{(n)}, \ldots, U_d^{(n)})$ is to pick d independent samples from the Haar measure on the group $\mathcal{U}_n$. We can also ask: *what does a random matrix range look like?* We carried out some preliminary work, and we think that using strong convergence results of Haagerup and Thorbjørnsen [45] and Collins and Male [27], we can show that $\mathcal{W}(U^{(n)}) \xrightarrow{n\to\infty} \mathcal{W}(u_f)$ for Haar unitary ensembles, and that $\mathcal{W}(X^{(n)}) \xrightarrow{n\to\infty} \mathcal{W}(s)$ for Wigner ensembles, where s is a free semicircular tuple [106] and convergence is in the Hausdorff metric, almost surely. Using [67] we can describe $\mathcal{W}(s)$ and $\mathcal{W}(u_f)$. I believe that we will be able to prove that the numerical range satisfies $\mathcal{W}_1(X^{(n)}) \xrightarrow{n\to\infty} \overline{\mathbb{B}}_{2,d}$ (almost surely), generalizing the known case $d = 2$ [26].

3.4. **Matrix ranges, operator structures and applications.** Building on [28, 66, 82, 84], in [83] we studied how matrix ranges determine the strucutre of operator tuples. For compact *compression minimal* tuples, the matrix range is a complete unitary invariant (this also holds for normals). We characterized compression minimal compact tuples: these are precisely the multiplicity free tuples which are rigid (i.e. have trivial Shilov boundary); this is equivalent to being *direct sum minimal* and *nonsingular*. The results required analysis of different kinds of extreme points, those mentioned above as well as *crucial matrix extreme points*, which we introduced.

I will study the extent to which matrix ranges determine the structure of operator systems. *Under what conditions does the matrix range determine a compression minimal tuple up to unitary equivalence?* I'll develop concrete and computational aspects of matrix convexity and use this to find effective criteria for determining when is a tuple *hyperrigid* (in the sense that the operator system it generates is hyperrigid [14]). It is clear that we cannot read off whether a tuple is *rigid* (in the sense that is Shilov boundary is trivial) from the matrix range, but as we saw above the rigidity of a tuple interacts with the matrix range (e.g., in the case of compact tuples it is implied by compression minimaility, which is formulated in terms of the matrix range), and I will continue to study exactly how.

3.4.1. *Application to Arveson's conjecture. Arveson's essential normality conjecture* [12, 36] can be described as follows: consider the tuple of coordinate multipliers $M_z = (M_{z_1}, \dots, M_{z_d})$ in $\mathcal{M}_d = \mathrm{Mult}(H^2_d)$, and let $I \lhd \mathbb{C}[z_1, \dots, z_d]$ be a homogeneous variety. The conjecture is that the compression T of M_z to the space $H^2_d \ominus I$ is *essentially normal*, in the sense that $T_iT_j^* - T_j^*T_i$ is compact for all i, j. In [63] we proved that the conjecture holds for a certain ideal I if and only if the corresponding operator tuple T is hyperrigid. We will study the matrix ranges $\mathcal{W}(T)$ and use our results to shed light on this conjecture.

3.4.2. *Matrix range questions.* Which matrix convex sets $\mathcal{S}$ are equal to $\mathcal{W}(A)$ for $A \in B(\mathcal{H})$, with $\dim \mathcal{H} = n < \infty$? Which matrix convex sets $\mathcal{S}$ are equal to $\mathcal{W}^{\min}(K)$ for a convex body $K \subseteq \mathbb{C}^d$? What about $\mathcal{W}^{\max}(K)$? In particular, what properties of an operator tuple T allow $\mathcal{W}(T)$ to be the $\mathcal{W}^{\max}$ or $\mathcal{W}^{\min}$ of a convex body? What is the minimal n such that $\mathcal{W}^{\max}_n(K) = \mathcal{W}^{\min}_n(K)$ implies that K is a simplex, for all K? And what convex bodies K are such that $\mathcal{W}^{\max}(K)$ is equal to $\mathcal{W}(A)$ for A acting on a finite dimensional space? (for $\mathcal{W}^{\min}(K)$ the answer is if and only if K is a polytope [43, 84]). What is the minimal dimensional space $\mathcal{H}$ such that $\mathcal{W}^{\max}(\overline{\mathbb{B}}_{2,d}) = \mathcal{W}(A)$ for $A \in B(\mathcal{H})^d$? We conjecture that $A = F$ works, where F is the canonical d-tuple of anticommuting selfadjoint unitaries acting on a 2^{d-1} dimensional space (true for $d = 2$ [54]). This would imply that every selfadjoint d-tuple A with numerical range in the unit ball has an *anticommuting* unitary dilation.

3.4.3. *Remark on finite dimensions.* Although we do not ignore it, we deliberately choose to mostly eschew Davidson and Kennedy's solution of adding infinite dimensional points. The infinite dimensional points certainly bring an advantage, as Davidson and Kennedy's clean results show. But when using nc sets that consist only of finite dimensional matrices, the theorems are more striking and maybe even more useful. This point of view meshes with a recent trend of highlighting finite dimensional phenomena [25, 48, 68, 69, 71] (see also [28, 40, 44, 84]).

3.5. **Algebras of bounded nc analytic functions on nc varieties.** Let $\mathcal{V}, \mathcal{W} \subset \mathbb{M}^d$ be bounded nc domains, and let $\mathfrak{V} \subseteq \mathcal{V}$ and $\mathfrak{W} \subseteq \mathcal{W}$ be nc analytic varieties. The problem that will be the engine of our research on nc function theory is: *when are $H^\infty(\mathfrak{V})$ and $H^\infty(\mathfrak{W})$ algebraically or (completelely) boundedly/isometrically isomorphic?* In this generality, this is likely to be hopeless. We can assume, at least at a first stage, that $\mathcal{V}$ and $\mathcal{W}$ are matrix convex. An important subproblem is $\mathcal{V} = \mathcal{W}$, that is, when both varieties are subvarieties of the same domain. The special case where $\mathcal{V} = \mathcal{W} = \mathfrak{B}_d$ has been studied by my collaborators and me [98, 99, 103]. In this case, $\mathfrak{V}$ is the space of finite dimensional WOT-continuous completely contractive representations of $H^\infty(\mathfrak{V})$, and $\widetilde{\mathfrak{V}}$ corresponds to the bounded ones. Our goal is to understand the representation spaces of the algebras we are studying in general. In the case of subvarieties of the ball, it is also true that $H^\infty(\mathfrak{V})$ is the quotient of $H^\infty(\mathfrak{B}_d)$ by the ideal of nc functions vanishing on $\mathfrak{V}$. It is not at all clear under what conditions this will hold true for general domains $\mathcal{V}$. Another important subproblem is when $\mathfrak{V} = \mathcal{V}$ and $\mathfrak{W} = \mathcal{W}$, that is, we are asking when are algebras of functions on domains are isomorphic. This leads to the question of classification of nc domains. Interesting work has been done on this question [7, 8, 15, 93], but not much.

We will also ask these question with regards to the algebras of continuous nc functions $A(\mathfrak{V})$. This raises interesting function-theoretic issues. The difficulty here is that first we must determine the nature of $A(\mathcal{V})$ — it is not even clear that $A(\mathcal{V})$ is the closure of polynomials in the supremum norm (for $\mathcal{V} = \mathfrak{B}_d$ it is true [98]). To approach all these problems we shall try to set things up in the context of nc RKHSs and multiplier algebras, making our algebras operator algebras in a natural way.

I conjecture that we will find reasonable conditions on a bounded nc domain $\mathcal{V}$ that will guarantee that for (reasonable) subvarieties $\mathfrak{V}, \mathfrak{W} \subseteq \mathcal{V}$, the algebras $H^\infty(\mathfrak{V})$ and $H^\infty(\mathfrak{W})$ are (completely) isometrically isomorphic if and only if there is a nc automorphism of $\mathcal{V}$ mapping $\mathfrak{W}$ onto $\mathfrak{V}$. Further I conjecture that $H^\infty(\mathfrak{V})$ and $H^\infty(\mathfrak{W})$ will be boundedly isomorphic if and only if there is nc bi-Lipschitz biholomorphism from $\widetilde{\mathfrak{W}}$ onto $\widetilde{\mathfrak{V}}$. Bi-Lipschitz with respect to what metric? In [98, 99, 103] we introduced a nc pseudohyperbolic metric on the similarity envelope $\widetilde{\overline{\mathfrak{B}}_d}$ of the closed ball by $\delta(X, Y) = \|\Phi_X - \Phi_Y\|$, where Φ_X is the evaluation functional defined on $A(\overline{\mathfrak{B}}_d)$ by $\Phi_X(f) = \tilde{f}(X)$ (here $\tilde{f}$ is the unique extension of f to $\widetilde{\overline{\mathfrak{B}}_d}$). Maybe that approach will work more generally. Perhaps we will find use in other approaches [19, 95]. The classification of nc domains can be carried out independently, and I plan to attack it as well.

3.5.1. *Open problems in the setting of the nc unit ball.* There are three open problems left to solve for the isomorphism problem for the algebras $H^\infty(\mathfrak{V})$ where $\mathfrak{V}$ is a subvariety of the nc ball $\mathfrak{B}_d$. In [99], it is proved that for homogeneous varieties $\mathfrak{V}, \mathfrak{W} \subseteq \mathfrak{B}_d$, the algebras $H^\infty(\mathfrak{V})$ and $H^\infty(\mathfrak{W})$ are boundedly isomorphic if and only if there is a linear map $A : \mathbb{C}^d \to \mathbb{C}^d$ mapping $\widetilde{\mathfrak{W}}$ bijectively onto $\widetilde{\mathfrak{V}}$, that is bi-Lipschitz w.r.t. the nc pseudohyperbolic distance. This begs the question: *given a linear map $A : \mathbb{C}^d \to \mathbb{C}^d$ mapping $\widetilde{\mathfrak{W}}$ bijectively onto $\widetilde{\mathfrak{V}}$, must A be bi-Lipschitz with respect to the nc pseudohyperbolic distance?* I conjecture this is true (of course!), but it is very diffucult, and proving it will contains Hartz's fantastic result [46] as a special case.

The second open problem in the ball is to remove the weak-$*$ continuity assumption from the non-homogneoeus case. This requires to adapt the Schwarz-Cartan type techniques that we used in the homogeneous case to the general case, a goal worthy in itself.

The third open problem is to improve the results in the the matrix unit ball in infinitely many variables $\mathfrak{B}_\infty$. For example, we basically have the results in the infinitely many variables case, under a weak-$*$ assumption, and we'd like to remove it (at least for homogeneous varieties). We also do not know whether isometric isomorphism and completely isometric isomorphism are equivalent in the infinite dimensional, homogeneous case. Analytic functions in infinitely many variables are very tricky but have some glaring advantages. For example, inspired the results in [72] it will be interesting to study the properties of the Bohr map (related of Dirichlet series) in the noncommutative setting. Also, if we wish to find a universal complete Pick nc kernel à la Agler-M$^{\text{c}}$Carthy [1], we should probably be looking at RKHSs of nc analytic functions in infinitely many variables ([72] notwithstanding).

3.6. **Developing nc function theory.** We already noted above the need to study classification of nc domains and develop Schwarz-Cartan type uniqueness results for non-homogeneous similarity envelopes. Here are a couple more problems. *Given a nc analytic function f in a nc open set, which vanishes on a nc variety $\mathfrak{V}$, is it true that f can be approximated (in various senses) by the ideal of all polynomials vanishing on $\mathfrak{V}$?* This question has an affirmative answer in the commutative homogeneous case [34], and solving it may shed light on the nonhomogeneous problem in the commutative setting as well. A closely related problem is *whether a nc analytic function on a nc subvariety of some nc open set can be extended to a nc analytic function on the open set.* The answer is "yes" in the case of an *algebraic* variety in a basic free open set [3]. The answer is also "yes" for an *analytic* nc variety in $\mathfrak{B}_d$, by the nc Pick property [18] (see also [98]). This problem can be approached with operator algebraic methods, in particular nc RKHS methods. Another possbile route of independent interest is to first develop rudiements nc function theory following the development of the theory of several complex variables, through notions of nc poloynomial/holomorphic convexity, nc pseudoconvexity, etc.

4. Conditions available for the research

The Technion provides me with all the physical conditions required for carrying out research. Currently the operator algebras group consists of two faculty members, a visiting professor, a postdoc and a PhD student. Our group has ties with other groups in Israel; we have local and national meetings. I request funding for a postdoc and two PhD students (most of my supervisees went on to do very nice things, so it's a good investment). I'll collaborate with all members of the group and coordinate the work. I'll have a critical mass for a seminar, studying the literature, and doing research. The students and postdocs will need computers, literature, and travel funds.

5. Pitfalls and alternative strategies

The abundance of research directions might become a pitfall. It might be difficult to coordinate the research between the different problems, or perhaps we will obtain scattered results that will not combine together to a coherent whole. On the other hand, my proposal has many alternative paths to take if research along one line gets stuck. I believe that in the end the results from both parts will turn out to be related. I expect that together with postdocs, students, and collaborators, I will make significant progress on all problems. I am suggesting several lines of research that can be carried out independently, difficulties in one direction will not stop us from making progress in others.

References

1. J. Agler and J.E. McCarthy, *Complete Nevanlinna–Pick Kernels*, J. Funct. Anal. 175 (2000), 111–124.
2. J. Agler and J.E. McCarthy, *Global holomorphic functions in several non-commuting variables*, Canad. J. Math. 67 (2015), 241-285.
3. J. Agler and J.E. McCarthy, *Pick Interpolation for free holomorphic functions*, Amer. J. Math. 137 (2015), 1685–1701.
4. J. Agler and J.E. McCarthy, *The implicit function theorem and free algebraic sets*, Trans. Amer. Math. Soc. 368 (2016), 3157–3175.
5. J. Agler, J.E. McCarthy and N.J. Young, *Non-commutative manifolds, the free square root and symmetric functions in two non-commuting variables*, Trans. London Math. Soc. 5 (2018), 132–183.
6. C.G. Ambrozie and A. Gheondea, *An interpolation problem for completely positive maps on matrix algebras: solvability and parametrization*, Linear and Multilinear Algebra 63 (2015), 826–851.
7. A. Arias and F. Latrémolière, *Isomorphisms of non-commutative domain algebras*, J. Operator Theory 66 (2011), 425–450.
8. A. Arias and F. Latrémolière, *Classification of noncommutative domain algebras*, C. R. Math. 370 (2012), 609–611.
9. W.B. Arveson, *Subalgebras of C^*-algebras*, Acta Math. 123 (1969), 141–224.
10. W.B. Arveson, *Subalgebras of C^*-algebras II*, Acta Math. 128 (1972), 271–308.
11. W.B. Arveson, *Subalgebras of C^*-algebras III: Multivariable operator theory*, Acta Math. 181 (1998), 159–228.
12. W.B. Arveson, *Quotients of standard Hilbert modules*, Trans. Amer. Math. Soc. 359 (2007), 6027–6055.
13. W.B. Arveson, *The noncommutative Choquet boundary*, J. Amer. Math. Soc. 21 (2008), 1065–1084.
14. W.B. Arveson, *The noncommutative Choquet boundary II: hyperrigidity*, Israel J. of Math. 184 (2011), 349–385.
15. M. Augat, J.W. Helton, I. Klep and S. McCullough, *Bianalytic maps between free spectrahedra*, Math. Ann. 371 (2018), 883–959.
16. J. Avron, P.H.M.v. Mouche and B. Simon, *On the measure of the spectrum for the almost Mathieu operator*, Commun. Math. Phys. 132 (1990), 103–118.
17. J.A. Ball, G. Marx and V. Vinnikov, *Noncommutative reproducing kernel Hilbert spaces*, J. Funct. Anal. 271 (2016), 1844–1920.
18. J.A. Ball, G. Marx and V. Vinnikov, *Interpolation and transfer-function realization for the noncommutative Schur-Agler class*, In Operator theory in different settings and related applications, pp. 23–116. Birkhäuser, Cham, 2018.
19. S.T. Belinschi and V. Vinnikov, *Noncommutative hyperbolic metrics*, preprint, arXiv:1707.09762.
20. J. Bellisard, *Lipshitz continuity of gap boundaries for Hofstadter-like spectra*, Commun. Math. Phys. 160 (1994), 599–613.
21. H. Bercovici, C. Foias, L. Kerchy and B. Sz.-Nagy, "Harmonic analysis of operators on Hilbert space", Universitext, *Springer, New York*, 2010.
22. A. Bluhm and I. Nechita, *Joint measurability of quantum effects and the matrix diamond*, J. Math. Phys. 59 (2018), 112202.
23. F.P. Boca, *Rotation C^*-Algebras and Almost Mathieu Operators*, The Theta Foundation, Bucharest, 2001.
24. A. Chefles, R. Jozsa and A. Winter, *On the existence of physical transformations between sets of quantum states*, Int. J. Quantum Inf. 2 (2004), 11–21.
25. D. Cohen, D. "Dilations of matrices", MSc. Thesis, Ben-Gurion University, 2015. arXiv:1503.07334.
26. B. Collins, P. Gawron, A.E. Litvak and K Życzkowski, *Numerical range for random matrices*, J. Math. Anal. Appl. 418 (2014), 516–533.
27. B. Collins and C. Male, *The strong asymptotic freeness of Haar and deterministic matrices*, Ann. Sci. Éc. Norm. Supér. 47 (2014), 147–163.
28. K.R. Davidson, A. Dor-On, O.M. Shalit and B. Solel, *Dilations, inclusions of matrix convex sets, and completely positive maps*, Int. Math. Res. Not. IMRN 2017, 4069–4130.

29. K.R. Davidson, M. Hartz and O.M. Shalit, *Multipliers of embedded discs*, Complex Anal. Oper. Theory 9 (2015), 287–321.
30. K.R. Davidson and M. Kennedy, *The Choquet boundary of an operator system*, Duke J. Math. 164 (2015), 2989–3004.
31. K.R. Davidson and M. Kennedy, *Noncommutative Choquet theory*, preprint, arXiv:1905.08436.
32. K.R. Davidson and D.R. Pitts, *Nevanlinna-Pick interpolation for non-commutative analytic Toeplitz algebras*, Integral Equations Operator Theory 31 (1998), 321–337.
33. K.R. Davidson and D.R. Pitts, *The algebraic structure of non-commutative analytic Toeplitz algebras*, Math. Ann. 311 (1998), 275–303.
34. K.R. Davidson, C. Ramsey and O.M. Shalit, *The isomorphism problem for some universal operator algebras*, Adv. Math. 228 (2011), 167–218.
35. K.R. Davidson, C. Ramsey and O.M. Shalit, *Operator algebras for analytic varieties*, Trans. Amer. Math. Soc. 367 (2015), 1121–1150.
36. R.G. Douglas, *A new kind of index theorem*, In: Analysis, Geometry and Topology of Elliptic Operators (Roskilde, Denmark, 2005) World Sci. Publishing, Singapore, 2006.
37. M. A. Dritschel and S. A. McCullough, *Boundary representations for families of representations of operator algebras and spaces*, J. Operator Theory 53 (2005), 159–167.
38. E.G. Effros and S. Winkler, *Matrix convexity: Operator analogues of the bipolar and Hahn-Banach theorems*, Journal of Functional Analysis 144 (1997), 117–152.
39. E. Evert, *Matrix convex sets without absolute extreme points*, Linear Algebra Appl. 537 (2018), 287–301.
40. E. Evert and J.W. Helton, *Arveson extreme points span free spectrahedra*, Math. Ann. 375 (2019), 629–653.
41. E. Evert, J.W. Helton, I. Klep and S. McCullough, *Extreme points of matrix convex sets, free spectrahedra and dilation theory*, J. Geom. Anal. 28 (2018) 1373–1408.
42. D.R. Farenick, *Extremal matrix states on operator systems*, J. London Math. Soc. 61 (2000), 885–892.
43. T. Fritz, T. Netzer and A. Thom, *Spectrahedral containment and operator systems with finite-dimensional realization*, SIAM J. Appl. Algebra Geom. 1 (2017), 556–574.
44. M. Gerhold and O.M. Shalit, *Dilations of q-commuting unitaries*, preprint, arXiv:1902.10362.
45. U. Haagerup and S. Thorbjørnsen, *A new application of random matrices:* $\mathrm{Ext}(C^*_{\mathrm{red}}(F_2))$ *is not a group*, Ann. of Math. 162 (2005), 711–775.
46. M. Hartz, *Topological isomorphisms for some universal operator algebras*, J. Funct. Anal. 263:11 (2012), 3564–3587.
47. M. Hartz, *On the isomorphism problem for multiplier algebras of Nevanlinna-Pick spaces*, Canad. J. Math. 69 (2017), 54–106.
48. M. Hartz and M. Lupini, *Dilation theory in finite dimensions and matrix convexity*, preprint, arXiv:1910.03549.
49. J.W. Helton, I. Klep and S. McCullough, *Proper analytic free maps*, J. Funct. Anal. 260 (2011), 1476–1490.
50. J.W. Helton, I. Klep and S. McCullough, *The convex Positivstellensatz in a free algebra*, Adv. Math. 231 (2012), 516–534.
51. J.W. Helton, I. Klep and S. McCullough, *The matricial relaxation of a linear matrix inequality*, Mathematical Programming 138 (2013), 401–445.
52. J.W. Helton, I. Klep, and S. McCullough, *Matrix convex hulls of free algebraic sets*, Trans. Amer. Math. Soc. **368** (2016), 3105–3139.
53. J.W. Helton, I. Klep and S. McCullough, *The Tracial Hahn-Banach Theorem, Polar Duals, Matrix Convex Sets, and Projections of Free Spectrahedra*, J. Eur. Math. Soc. 19 (2017) 1845–1897.
54. J.W. Helton, I. Klep, S. McCullough and M. Schweighofer, *Dilations, Linear Matrix Inequalities, the Matrix Cube Problem and Beta Distributions*, Mem. Amer. Math. Soc. 257 vol. 1232 (2019).
55. J.W. Helton and S. McCullough, *Every convex free basic semi-algebraic set has an LMI representation*, Annals of Mathematics 176 (2012), 979–1013.
56. D.R. Hofstadter, *Energy levels and wave functions of Bloch electrons in rational and irrational magnetic fields*, Phys. Rev. B. 14 (1976), 2239–2249.
57. B. Huber and T. Netzer, *A note on non-commutative polytopes and polyhedra* preprint, arXiv:1809.00476.

58. N. Johnston, D.W. Kribs, V.I. Paulsen and R. Pereira, *Minimal and maximal operator spaces and operator systems in entanglement theory*, J. Funct. Anal. 260 (2011), 2407–2423.
59. M.T. Jury and R.T.W. Martin, *Non-commutative Clark measures for the free and abelian Toeplitz algebras*, J. Math. Anal. Appl. 456 (2017), 1062–1100.
60. M.T. Jury and R.T.W. Martin, *Fatou's theorem for non-commutative measures*, preprint, arXiv:1907.09590.
61. E.T.A. Kakariadis and O.M. Shalit, *On operator algebras associated with monomial ideals in non-commuting variables*, J. Math. Anal. Appl. 472 (2019), 738–813.
62. D. Kalyuzhnyi-Verbovetskii and V. Vinnikov, "Foundations of Noncommutative Function Theory". Vol. 199. American Mathematical Soc., 2014.
63. M. Kennedy and O.M. Shalit, *Essential normality, essential norms and hyperrigidity*, J. Funct. Anal. 268 (2015), 2990–3016.
64. M. Kerr, J.E. McCarthy and O.M. Shalit, *On the isomorphism question for complete Pick multiplier algebras*, Integral Equations Operator Theory 76 (2013), 39–53.
65. I. Klep and M. Schweighofer, *Connes' embedding conjecture and sums of hermitian squares*, Adv. Math. 217 (2008), 1816–1837.
66. T.-L. Kriel, *An introduction to matrix convex sets and free spectrahedra*, preprint, arXiv:1611.03103v6.
67. F. Lehner, *Computing norms of free operators with matrix coefficients*, Amer. J. Math. 121 (1999), 453–486
68. J. Levick and R.T.W. Martin, *Matrix N-dilations of quantum channels*, preprint, arXiv:1808.04677.
69. E. Levy and O.M. Shalit, *Dilation theory in finite dimensions: the possible, the impossible and the unknown*, Rocky Mountain J. Math. 44 (2014), 203–221.
70. C.K. Li and Y.T. Poon, *Interpolation by completely positive maps*, Linear and Multilinear Algebra 59 (2011), 1159–1170.
71. J.E. McCarthy and O.M. Shalit, *Unitary N-dilations for tuples of commuting matrices*, Proc. Amer. Math. Soc. 14 (2013), 563–571.
72. J.E. McCarthy and O.M. Shalit, *Spaces of Dirichlet series with the complete Pick property*, Israel J. Math. 220 (2017), 509–530.
73. J.E. McCarthy and R. Timoney, *NC-automorphisms of nc-bounded domains*, Proc. Roy. Soc. Edinburgh Sect. A 146 (2016), 1037–1045.
74. P.S. Muhly and B. Solel, *An algebraic characterization of boundary representations*, in: "Nonselfadjoint operator algebras, operator theory, and related topics", Birkhäuser, Basel, 1998. 189–196.
75. P.S. Muhly and B. Solel, *Hardy algebras, W*-correspondences and interpolation theory*, Math. Ann. 330 (2004), 353–415.
76. P.S. Muhly and B. Solel, *Hardy algebras associated with W*-corsrespondences (point evaluation and Schur class functions*, In Operator theory, systems theory and scattering theory: multidimensional generalizations, volume 157 of Oper. Theory Adv. Appl., pages 221–241. Birkhäuser, Basel, 2005
77. P.S. Muhly and B. Solel, *Representations of Hardy algebras: absolute continuity, intertwiners, and superharmonic operators*, Integral Equations Operator Theory 70 (2011), 151–203.
78. P.S. Muhly and B. Solel, *Tensorial function theory: From Berezin transforms to Taylor's Taylor series and back*, Integral Equations Operator Theory 76 (2013), 463–508.
79. P.S. Muhly and B. Solel, *Matricial function theory and weighted shifts*, Integral Equations Operator Theory 84 (2016), 501–553.
80. J.E. Pascoe. *The inverse function theorem and the Jacobian conjecture for free analysis*, Math. Z. 278 (2014), 987–994.
81. J.E. Pascoe, *An entire free holomorphic function which is unbounded on the row ball*, preprint, arXiv:1908.06753.
82. B. Passer, *Shape, scale, and minimality of matrix ranges*, Trans. Amer. Math. Soc. (2019).
83. B. Passer and O.M. Shalit, *Compressions of compact tuples*, Linear Algebra Appl. 564 (2019), 264–283.
84. B. Passer, O.M. Shalit and B. Solel, *Minimal and maximal matrix convex sets*, J. Funct. Anal. 274 (11), 3197–3253.
85. V.I. Paulsen, *Representations of function algebras, abstract operator spaces, and Banach space geometry*, J. Funct. Anal. 109 (1992), 113–129.
86. V Paulsen, "Completely bounded maps and operator algebras", Cambridge University Press, 2002.

87. V.I. Paulsen, I.G. Todorov and M. Tomforde, *Operator system structures on ordered spaces* Proc. Lond. Math. Soc. 102 (2011), 25–49.
88. G. Popescu, *Isometric dilations for infinite sequences of noncommuting operators*, Trans. Amer. Math. Soc. 316 (1989), 523–536.
89. G. Popescu, *Von Neumann inequality for* $(B(\mathcal{H})^n)_1$, Math. Scand. 68 (1991), 292–304.
90. G. Popescu, *Operator Theory on Noncommutative Varieties*, Indiana Univ. Math. J. 56 (2006), 389–442.
91. G. Popescu, *Free holomorphic functions on the unit ball of* $B(\mathcal{H})^n$, J. Funct. Anal. 241 (2006), 268–333.
92. G. Popescu, *Free holomorphic automorphisms of the unit ball of* $B(\mathcal{H})^n$, J. Reine Angew. Math. 638 (2010), 119–168.
93. G. Popescu, *Free biholomorphic classification of noncommutative domains*, Int. Math. Res. Not. IMRN 2011, 784–850.
94. G. Popescu, *Holomorphic automorphisms of noncommutative polyballs*, J. Operator Theory 76 (2016), 387–448.
95. G. Popescu, *Hyperbolic geometry on noncommutative polyballs*, J. Math. Anal. Appl. 456 (2017), 576–607.
96. G. Popescu, *Free holomorphic functions on polydomains*, in: "Operator Theory: Themes and Variations", Theta Series in Advanced Mathematics, 2018, 157–186.
97. G. Salomon and O.M. Shalit, *The isomorphism problem for complete Pick algebras: a survey*, Oper. Theory Adv. Appl., Vol. 255. Birkhäuser, 166–198, 2016.
98. G. Salomon, O.M. Shalit and E. Shamovich, *Algebras of bounded noncommutative analytic functions on subvarieties of the noncommutative unit ball*, Trans. Amer. Math. Soc. 370 (2018), 8639–8690.
99. G. Salomon, O.M. Shalit and E. Shamovich, *Algebras of noncommutative functions on subvarieties of the noncommutative ball: the bounded and completely bounded isomorphism problem*, preprint (2018) arXiv:1806.00410. To appear in J. Funct. Anal.
100. O.M. Shalit, *Operator theory and function theory in Drury-Arveson space and its quotients*, in "Handbook of Operator Theory", Springer References, Springer, 1125–1180, 2015.
101. O.M. Shalit, *The complex matrix cube problem - new results from summer projects*, blog post, in "Noncommutative Analysis", https://noncommutativeanalysis.com, October 2019.
102. O.M. Shalit and B. Solel, *Subproduct systems*, Documenta Math. 14 (2009), 801–868.
103. E. Shamovich, *On fixed points of self maps of the free ball*, J. Funct. Anal. 275 (2018), 422–441.
104. J.L. Taylor, *A general framework for a multi-operator functional calculus*, Adv. Math. 9 (1972), 183–252.
105. J.L. Taylor, *Functions of several noncommuting variables*, Bull. Amer. Math. Soc. 79 (1973), 1–34.
106. D.V. Voiculescu, K. J. Dykema and A. Nica "Free random variables", CRM Monographs Series, No. 1. American Mathematical Soc., 1992.
107. C. Webster and S. Winkler, *The Krein-Milman theorem in operator convexity*, Trans. Amer. Math. Soc. 351 (1999), 307–322.
108. G. Wittstock, *On matrix order and convexity*, in: "Functional Analysis: Surveys and Recent Results," Math. Studies, Vol. 90, pp. 175–188, North-Holland, Amsterdam, 1984.
109. A. Zalar. *Operator positivestellensatze for noncommutative polynomials positive on matrix convex sets*, J. Math. Anal. Appl. 445 (2017), 32–80.

Time schedule and work-plan

Objective	Beginning	End

Explanatory Notes:

This is a project in pure mathematics and if I were to write something down here it would be just to fill something in. Honestly I don't know any pure mathematician who works according to a work plan.

Budget details

Personnel

Name (last, first)	Role in project	% time devoted	Salaries (in NIS)			
			1st year	2nd year	3rd year	4th year
Shalit Orr Moshe	PI	25	0	0	0	0
postdoc	RA 1	100	144,000	144,000	144,000	144,000
PhD student	RA 2	50	40,000	40,000	40,000	40,000
PhD student	RA 3	50	40,000	40,000	40,000	40,000
Total Personnel			**224,000**	**224,000**	**224,000**	**224,000**

Justification for requested Personnel:

My project is a collection of problems from several different frontiers in noncommutative analysis, and it requires several people working in tandem to advance. It is very hard for a PhD student to work on both the "real" and the "complex" sides of the project. Each student will work on a project on one of the sides. I will try to hire an excellent postdoc who has skills to complement my own (several complex variables, free probability or random matrix theory), who will contribute in strategic junctions. I have a very good record of leading and coordinating a research group of 2-3 people and training junior researchers.

Supplies & Materials

Item	Requested sums (in NIS)			
	1st year	2nd year	3rd year	4th year
Total Supplies & Materials	**0**	**0**	**0**	**0**

Justification for requested Supplies & Materials:

Services

Item	Requested sums (in NIS)			
	1st year	2nd year	3rd year	4th year
Total Services	**0**	**0**	**0**	**0**

Justification for requested Services:

Other Expenses

Item	Requested sums (in NIS)			
	1st year	2nd year	3rd year	4th year
Participation of research assistants in international conferences	6,000	6,000	6,000	6,000
Total Other Expenses	**6,000**	**6,000**	**6,000**	**6,000**

Justification for requested Other Expenses:

Research assistants (PhD students, postdocs) need funding to go to conferences and workshops on a yearly basis. This is important for several reasons:

1. Professional development - to learn new and cutting edge results.
2. Networking - meeting senior and junior colleagues from around the world, who may help them find new opportunities to advance their careers.
3. Dissemination - the postdoc and graduate students will give lectures on our results and advertise them, thereby increasing the impact of our research.

Computers

Item	Requested sums (in NIS)			
	1st year	2nd year	3rd year	4th year
Personal computer for the researcher	20,000	0	0	0
Personal computer for students/research assistants	12,000	6,000	0	0
Software	0	0	0	0
Peripherals	0	0	0	0
Cloud computing	0	0	0	0
Total Computers	**32,000**	**6,000**	**0**	**0**

Justification for requested Computers:

I need to buy a quality laptop and a desktop, essential for my work. The desktop is required to work comfortably in my office - my current one is old. I will use the laptop for working while traveling.

The graduate students will also need laptops, and the postdoc will need a desktop (work station).

Miscellaneous

Item	Requested sums (in NIS)			
	1st year	2nd year	3rd year	4th year
Internet Connection (office/lab only)	0	0	0	0
Photocopies and office supplies	1,000	1,000	1,000	1,000
Memberships in scientific associations	350	350	350	350
Publication charges in scientific journals (including editing and translation)	0	0	0	0
Professional literature	1,500	1,500	1,500	1,500
Total Miscellaneous	**2,850**	**2,850**	**2,850**	**2,850**

Justification for requested Miscellaneous:

I am asking for a modest budget for basic supplies, membership in associations (IMU and EMU) and books.

Budget Summary

	Requested sums (in NIS)			
	1st year	2nd year	3rd year	4th year
Personnel	224,000	224,000	224,000	224,000
Supplies & Materials	0	0	0	0
Services	0	0	0	0
Other Expenses	6,000	6,000	6,000	6,000
Computers	32,000	6,000	0	0
Miscellaneous	2,850	2,850	2,850	2,850
Overhead	45,024	40,604	39,584	39,584
Equipment (no overhead on this item)	0			
Total budget	**309,874**	**279,454**	**272,434**	**272,434**
Annual average	**283,549**			
International Cooperation (including overhead)	0			

Curriculum Vitae

Name: Shalit Orr Moshe

A. Academic Background

Date (from-to)	Institute	Degree	Area of specialization
2001-2003	Technion	BSc	Mathematics
2003-2005	Technion	MSc	Mathematics
2005-2009	Technion	PhD	Mathematics (Operator algebras and operator theory) Advisor: Prof. Baruch Solel

B. Previous Employment

Date (from-to)	Institute	Title	Research area
2018-present	Technion - Israel Institute of Technology	Associate Professor	Mathematics
2014-2018	Technion - Israel Institute of Technology	Assistant Professor	Mathematics
2011-2014	Ben Gurion University	Senior Lecturer	Mathematics
2009-2011	University of Waterloo	Postoctoral Fellow	Mathematics
2004-2006	Rafael Ltd.	Researcher	Computer vision/image processing
2002-2003	Rafael Ltd.	Researcher	Computer vision/image processing

C. Grants and Awards Received Within The Past Five Years

Date (from-to)	Research Topics	Funding Organization	Total (in NIS)
2016-2020	Operator Theory, Operator Algebras	Israel Science Foundation	920000
Comments			
2015-2016	Operator algebras, completely positive maps	The Gerald Schwartz & Heather Reisman Foundation (Waterloo – Technion)	115000
Comments	Jointly with Prof. Ken Davidson (Waterloo) and Prof. Baruch Solel (Technion)		

O.M. SHALIT — LIST OF PUBLICATIONS

1. Publications

1.1. Theses.

(1) O.M. Shalit, "Guided dynamical systems and applications to functional and partial differential equations", Msc. Thesis, 2005. Supervisor: Boris Paneah.
(2) O.M. Shalit, "Product systems, subproduct systems, and dilation theory of completely positive semigroups", PhD. Thesis, 2009. Supervisor: Baruch Solel.

1.2. Books.

(1) O.M. Shalit, "A First Course in Functional Analysis", CRC Press, 240 pages, 2017.

1.3. Chapters in books.

(1) O.M. Shalit, *Operator theory and function theory in Drury-Arveson space and its quotients*, in **Handbook of Operator Theory**, Springer References, Springer, 1125–1180, 2015.

1.4. Papers in professional peer reviewed journals (accepted).

(1) O.M. Shalit, *On the Overdeterminedness of a Class of Functional Equations*, **Aequationes Math.**, Vol. 74, No. 3 (2007), 242–248.
(2) O.M. Shalit, *E_0-dilation of strongly commuting CP_0-semigroups*, **J. Funct. Anal.**, Vol. 255, No. 1 (2008), 46–89. (Corrigendum appeared in Vol. 258, No. 3 (2010), 1068-1069).
(3) O.M. Shalit, *What type of dynamics arise in E_0-dilations of commuting quantum Markov semigroups?*, **Infin. Dimens. Anal. Quantum Probab. Relat. Top.**, Vol. 11, No. 3 (2008), 393–403.
(4) E. Levy and O.M. Shalit, *Continuous extension of a densely parameterized semigroup*, **Semigroup Forum**, Vol. 78, No. 2 (2009), 276–284.
(5) O.M. Shalit and B. Solel, *Subproduct systems*, **Doc. Math.** Vol. 14 (2009), 801–868.
(6) O.M. Shalit, *Conjugacy of P-configurations and nonlinear solutions to a certain conditional Cauchy equation*, **Banach J. Math. Anal.**, Vol. 3, No. 1 (2009), 28–35.
(7) O.M. Shalit, *Representing a product system representation as a contractive semigroup and applications to regular isometric dilations*, **Canad. Math. Bull.**, Vol. 53 (2010), 550–563.
(8) D. Markiewicz and O.M. Shalit, *Continuity of CP-semigroups in the point-strong topology*, **J. Operator Theory**, Vol. 64, No.1 (2010), 149–154.
(9) O.M. Shalit, *E-dilation of strongly commuting CP-semigroups (the nonunital case)*, **Houston J. Math.**, Vol. 35 No. 1 (2011), 203–232.

(10) O.M. Shalit, *Stable polynomial division and essential normality of graded Hilbert modules*, **J. Lond. Math. Soc.**, Vol. 83 Issue 2 (2011), 273–289.

(11) O.M. Shalit and M. Skeide, *Three commuting, unital, completely positive maps that have no minimal dilation*, **Integral Equations Operator Theory**, Vol. 71 Issue 1 (2011), 55–63.

(12) K.R. Davidson, C. Ramsey and O.M. Shalit, *The isomorphism problem for some universal operator algebras*, **Adv. Math.**, Vol. 228 (2011), 167–218.

(13) O.M. Shalit, *Three remarks on a question of Aczél*, **Aequationes Math.**, Vol. 84, Issue 3 (2012), 201–205.

(14) M. Kennedy and O.M. Shalit, *Essential normality and the decomposability of algebraic varieties*, **New York J. Math.**, Vol. 18 (2012), 877–890.

(15) J.E. McCarthy and O.M. Shalit, *Unitary N-dilations for tuples of commuting matrices*, **Proc. Amer. Math. Soc.**, Vol. 141 (2013), 563–571.

(16) M. Kerr, J.E. McCarthy and O.M. Shalit, *On the isomorphism question for complete Pick multiplier algebras*, **Integral Equations Operator Theory**, Vol. 76 Issue 1 (2013), 39–53.

(17) E. Levy and O.M. Shalit, *Dilation theory in finite dimensions: the possible, the impossible and the unknown*, **Rocky Mountain J. Math.**, Vol 44 Issue 1 (2014), 203–221.

(18) S. Pal and O.M. Shalit, *Spectral sets and distinguished varieties in the symmetrized bidisc*, **J. Funct. Anal.**, Vol. 266 Issue 9 (2014), 5779–5800.

(19) K.R. Davidson, M. Hartz and O.M. Shalit, *Multipliers of embedded discs*, **Complex Anal. Oper. Theory**, Vol. 9 (2015), 287–321 (Erratum appeared in Vol. 9, 323–327).

(20) K.R. Davidson, C. Ramsey and O.M. Shalit, *Operator algebras for analytic varieties*, **Trans. Amer. Math. Soc.**, Vol. 367 Issue 2 (2015), 1121–1150.

(21) M. Kennedy and O.M. Shalit, *Essential normality, essential norms and hyperrigidity*, **J. Funct. Anal.**, Vol. 268 Issue 10 (2015), 2990–3016.

(22) J.E. McCarthy and O.M. Shalit, *Spaces of Dirichlet series with the Complete Pick property*, **Israel J. Math.**, Vol. 220 No. 2 (2017), 509–530.

(23) K.R. Davidson, A. Dor-On , O.M. Shalit and B. Solel, *Dilations, inclusions of matrix convex sets, and completely positive maps*, **Int. Math. Res. Not. IMRN.**, Vol. 2017 No. 13, (2017), 4069–4130.

(24) S. Biswas and O.M. Shalit, *Stable division and essential normality: the non-homogeneous and quasi homogeneous cases*, **Indiana Univ. Math. J.**, Vol. 67 No. 1 (2018), 169–185.

(25) B. Passer, O.M. Shalit and B. Solel, *Minimal and maximal matrix convex sets*, **J. Funct. Anal.**, Vol. 274 (2018), 3197–3253.

(26) G. Salomon, O.M. Shalit and E. Shamovich, *Algebras of bounded noncommutative analytic functions on subvarieties of the noncommutative unit ball*, **Trans. Amer. Math. Soc.**, Vol. 370 No. 12 (2018), 8639–8690.

(27) E.T.A. Kakariadis and O.M. Shalit, *On operator algebras associated with monomial ideals in noncommuting variables*, **J. Math. Anal. Appl.**, Vol. 472 No. 1 (2019), 738–813.
(28) B. Passer and O.M. Shalit, *Compressions of compact tuples*, **Linear Algebra Appl.**, Vol. 564 (2019), 264–283.
(29) G. Salomon, O.M. Shalit and E. Shamovich, *Algebras of noncommutative functions on subvarieties of the noncommutative ball: the bounded and completely bounded isomorphism problem*, to appear in **J. Funct. Anal.**, 45 pages.

1.5. **Submitted papers.**

(1) M. Gerhold and O.M. Shalit, *Dilations of q-commuting unitaries*, 16 pages, submitted.

1.6. **Refereed papers in conference proceedings.**

(1) G. Salomon and O.M. Shalit, *The isomorphism problem for complete Pick algebras: a survey*, in **Oper. Theory Adv. Appl.**, Vol. 255. Birkhäuser, 166–198, 2016.

1.7. **Other publications.**

(1) O.M. Shalit, *A sneaky proof of the maximum modulus principle*, **Amer. Math. Monthly**, Vol. 120, no. 4 (2013), 359–362.

Current ISF application No.: 431/20
PI Name: Orr Moshe Shalit
Research title: Noncommutative analysis, dilation theory and operator algebras

Objectives, results, achievements & list of publications of a Recent ISF grant

Recent ISF grant No. 195/16
Research title: Operator-algebraic geometry in noncommutative analysis

Objectives and results

The objectives of my ISF funded project (grant no. 195/16) were the following:

1. To identify algebras of bounded noncommutative (nc) analytic functions on nc domains and varieties as operator algebras and study their structure. Conversely, to represent operator algebras as algebras of nc functions and classify the algebras up to isomorphism by the geometric properties of the nc varieties on which they live.
2. To study nc reproducing kernel Hilbert spaces.
3. To investigate approximation and extension problems for nc analytic functions on nc varieties. In the same vein, to consider noncommutative versions of the Nullstellensatz.
4. To use the results in the nc setting to shed light on the classical (commutative) case.
5. To apply methods of matrix convexity and noncommutative analysis to the interpolation problem for unital completely positive (UCP) maps, and to apply the results to study the existence and uniqueness of UCP maps with certain properties between operator spaces. Lastly, to apply these results to noncommutative Choquet theory in operator algebras.

Objectives 1 – 4 were all treated with considerable success in the two long papers [6] and [7], which were joint work with my postdoc Eli Shamovich and my PhD student Guy Salomon.

In [6] we identified the algebras of bounded nc functions on the nc unit ball and on subvarieties. Let *B* denote the nc unit ball (in d dimensions, where d will remain implicit) and let *H(B)* denote the algebra of bounded nc analytic functions on *B*. We showed canonical identifications between the algebra *H(B)* of bounded nc analytic functions on the nc unit ball, between *the non-commutative analytic Toeplitz algebra* studied by Davidson-Pitts and by Popescu, between (a special case of) the *noncommutative Hardy algebra* of Muhly-Solel. We also showed a canonical equivalence with the *nc multiplier algebra* of the *nc Drury-Arveson space* – a *nc reproducing kernel Hilbert space* in the sense of Ball-Marx-Vinnikov. We also gave a new proof of the fact that the underlying nc reproducing kernel Hilbert space is a *complete Pick space*.

To describe further results we need a bit of notation. Let *V* be a *nc subvariety* of *B*, and let *H(V)* be the algebra of bounded nc analytic functions on *V*. We identified *H(V)* with several operator algebras of interest, including the complete quotient *H(B)/J* of *H(B)* by the closed ideal *J* consisting of functions that vanish on *V*, as well as a *nc multiplier algebra* on a *nc reproducing Hilbert space* of nc functions on *V*. Further, when *V* is a homogeneous variety, then we showed that *H(V)* can be identified with the operator algebra associated with a *subproduct system.*

After finishing with the various identifications and connections (parts of Objectives 1 and 2), we moved on. Next, we proved that *H(V)* is completely isometrically isomorphic to *H(W)* if and only if there is a nc biholomorphism between *V* and *W*. For homogeneous varieties, we showed that this is equivalent to that *V* and *W* are conformally equivalent. This addresses Objective 1 in the case of completely isometric isomorphism. In passing we also addressed Objective 3 by developing a nc

Nullstellensatz applied in the proof, and we really were able to clarify somewhat the commutative case, in particular by identifying the multipliers on the commutative Drury-Arveson space as the bounded nc analytic functions on the "commutative ball".

In paper [7] we treated the classification problem in the setting of bounded and completely bounded isomorphism. The main theorem can be stated as follows: if *V* and *W* are homogeneous varieties, then *H(V)* is boundedly isomorphic to *H(W)* if and only if the *similarity envelopes* of *V* and *W* are biholomorphic via a bi-Lipschitz map. For this we needed to develop many tools in nc function theory, and we got dividends for Objectives 1 – 4 and beyond that. Of special interest is our novel analysis of the *similarity envelopes of varieties*, in particular our *nc Cartan uniqueness theorem* and *nc Schwarz lemma* for similarity envelopes, which are *unbounded* nc domains.

The results I obtained towards Objective 5 were quite surprising and have opened up some new research avenues for me [1,2,3,4,5]. In [1] the interpolation problem for UCP maps was shown to be equivalent to a containment problem for matrix ranges, and in tackling it we obtained some exciting results on minimal and maximal matrix convex sets lying over a scalar convex set. Most interestingly, we established a connection between containment of matrix convex sets and the existence of a normal dilation. In [5] we continued the study of minimal and maximal matrix convex sets. We found sharp dilation constants for all Lp balls, and in particular we proved that every d-tuple of self-adjoints contractions has a normal dilation with elements of norm at most square root of d. Another major result of [5] is the characterization of the simplex as the unique convex set, for which the minimal and maximal matrix convex sets over it coincide.

The above dilation results raised an interesting question: given a *d*-tuple of contractions *A*, what is the smallest constant *C* such that *A* has a normal dilation of norm *C*? In an attempt to answer this question, we studied the dilation constants for *q*-commuting unitaries [2]. This work found a very surprising application to the continuity of the spectrum of the almost Mathieu operator from mathematical physics.

In [4] we proved that the matrix range of a *fully compressed* tuple of compact operators is a unitary equivalence invariant, and we studied the connection of this notion to those of *minimality* and *singularity*. This urged us to understand what a do matrix ranges look like. We conjecture [3] that the matrix ranges of large random matrices converge almost surely in the Hausdorff metric to a well understood object (e.g., the matrix range of free semi-circulars). All this is progress on the problem of understanding operators and the operator systems they generate in terms of the matrix range and its various "boundaries", which is Objective 5.

List of publications

1. K.R. Davidson, A. Dor-On, B. Solel and O.M. Shalit. Dilations, inclusions of matrix convex sets, and completely positive maps, *International Mathematics Research Notices* Vol. 2017 Issue 13 (2017), 4069-4130.
2. M. Gerhold and O.M. Shalit. Dilations of q-commuting unitaries, *submitted, 17 pages*.
3. M. Gerhold and O.M. Shalit. The matrix range of a d-tuple of random matrices, *work in progress.*
4. B. Passer and O.M. Shalit. Compressions of compact tuples, *Linear Algebra Appl.* Vol. 564 (2019), 264-283.
5. B. Passer, O.M. Shalit and B. Solel. Minimal and maximal matrix convex sets, *J. Funct. Anal.* Vol 274 (2018), 3197-3253.
6. G. Salomon, O.M. Shalit and E. Shamovich. Algebras of bounded noncommutative analytic functions on subvarieties of the noncommutative unit ball, *Trans. Amer. Math. Soc.* Vol. 370 (2018), 8639-8690.
7. G. Salomon, O.M. Shalit and E. Shamovich. Algebras of noncommutative functions on subvarieties of the noncommutative ball: the bounded and completely bounded isomorphism problem, *to appear in J. Funct. Anal., 45 pages.*