The pathway to NC function theory
A research story

1 Introduction

This document contains some excerpts from Part B2 of my ERC grant proposal. The area of NC function theory is not as widely recognized as some other areas competing for grants, I therefore thought that it would be interesting for some readers if I told the mathematical story of how I was led to enter this area. My proposal ended up not being funded, and I thought that it might be of use to somebody out there if I made the expository parts of my proposal available online. The first section of this document is a brief overview of NC function theory and of the objects of study. People familiar with NC function theory can skim the first section and proceed quickly to the second section, which contains the account that describes how I was led into this field.

1.1 Motivation

A pervasive theme in operator algebra, since the early development of the field, has been the construction of operator algebras that encode a particular mathematical object, and the subsequent study of the relationship between the algebra and the underlying structure. Examples include the von Neumann algebra of a group or of a measurable action on a space, and graph C*-algebras. The quintessential example is the algebra $C(X)$ of continuous functions on a compact Hausdorff space. Gelfand taught us that $C(X) \cong C(Y)$ as algebras if and only if X is homeomorphic to Y. This, in itself, is a beautiful example of a categorical equivalence, but combined with Gelfand's characterization of commutative C*-algebras, it suggests something deeper: every operator algebra can be viewed as an algebra of "functions" on some "space", and opens the door for the paradigms of noncommutative analysis and noncommutative geometry.

This research proposal is situated within this tradition. At its core lies the relationship between an algebra and the space on which it is defined. The classes of algebras that I have been drawn to study consist of bounded noncommutative functions on subvarieties of operator balls. These algebras are neither as widely known as the examples above nor as thoroughly studied. In this proposal I aim to demonstrate that they merit significant attention and that they are linked to a compelling set of fundamental problems at the intersection of functional analysis, operator theory, function theory and complex geometry.

The algebra $H^\infty(\mathbb{D})$ of bounded analytic functions on the unit disc is the largest function algebra that supports a functional calculus for all operators $T \in B(H)$ with $\|T\| < 1$. A detailed understanding of $H^\infty(\mathbb{D})$ has yielded profound operator-theoretic results [10, 29]. Extending this framework to d-tuples of noncommuting operators, for which classical functional calculus is insufficient, leads one to consider certain completions of the algebra of free polynomials in d noncommuting variables and to a family of new function algebras: the algebras of bounded NC functions on a d-dimensional NC ball. There is not one single ball, but rather a distinct NC ball $\mathbb{B}_E$ and its corresponding algebra $H^\infty(\mathbb{B}_E)$ for every operator space E. Pioneering work by Agler–McCarthy, Popescu, and others indicates great potential, but these algebras and their quotients remain largely unexplored. I propose a systematic study of NC function algebras, with the central goal of elucidating the precise relationship between their algebraic structure and the underlying geometry.

Algebras of bounded NC functions on NC domains and their subvarieties are compelling for several additional reasons. First, because they form a rich class of tangible operator algebras that are amenable to analysis in concrete terms. Second, this class of algebras contains a wide spectrum of operator algebras that have arisen in disparate parts of functional analysis and independently of NC function theory. For example in [34, 35] the class of algebras of bounded NC functions on subvarieties of the row ball was shown to include multiplier algebras of complete Pick spaces [13, 14, 16] as well as tensor algebras of subproduct systems [24, 38]. Representing operator algebras as algebras of NC functions provides tractable invariants, enlightening insights, and new results (for example, applications to isomorphisms of subproduct system algebras in [34] or to the problem of quasi-inner automorphisms of Davidson–Pitts's noncommutative analytic Toeplitz algebras $\mathfrak{L}_d$ in [35]). Third, studying algebras of bounded NC functions, and in particular classifying them, has driven the discovery of purely NC function-theoretic and geometric theorems; for example: maximum modulus principle and Nullstellensatze [34], NC Schwarz lemma and spectral Cartan uniqueness theorem [35], fixed point and iteration theoretic results [9, 39], and the clarification of the notion of uniform continuity [17]. Recently, we have observed that these algebras provide concrete representations of universal operator algebras generated by an operator space and its quotients [36, 37], and this suggests deep interactions between operator space theory and NC analytic function theory.

1.2 Definitions and notation

Noncommutative (NC) analytic functions go back to Taylor's general theory [40, 41], and the independent theory by Voiculescu who, significantly, applied it to free probability [42, 43]. The theory has been developed by several groups of researchers, coming from different subfields of functional analysis, and motivated by a variety of applications in systems theory and control, operator theory, function theory, and algebra. We will work within the modern approach of Kaliuzhnyi-Verbovetskyi and Vinnikov [26], Helton–Klep–McCullough [19], Agler-McCarthy [3], Ball–Bolotnikov [7], Popescu [31, 32, 33] and others [4, 6, 8, 22, 21, 30].

NC sets and NC domains and operator balls

Let M_n denote the $n\times n$ matrices over $\mathbb{C}$. For an operator space E, we let $M_n(E) = M_n \otimes E$ and define $\mathbb{M}(E) = \cup_{n=1}^{\infty} M_n(E)$. We remind the reader that an operator space E is just a subspace of a C*-algebra with the inherited norm. This implies that $M_n(E)$ too is an operator space. We sometimes identify $E = \mathbb{C}^d$ equipped with a particular operator space structure (i.e. a family of matrix norms), in which $M_n(\mathbb{C}^d) = M_n^d$ is the set of all d-tuples of scalar matrices, and we then write $\mathbb{M}^d$ for $\mathbb{M}(\mathbb{C}^d) = \cup_{n=1}^{\infty} M_n^d$. A subset $\Omega \subseteq \mathbb{M}(E)$ is said to be an *NC set* if it is closed under direct sums. We write Ω_n or $\Omega(n)$ for $\Omega \cap M_n(E)$, so $\Omega = \cup_{n=1}^{\infty}\Omega_n$. The *similarity envelope* $\widetilde{\Omega}$ of an NC set Ω consists of all tuples $S^{-1}XS := (S^{-1}X_1S,\ldots,S^{-1}X_dS)$ where $X\in\Omega$, that is

$$\widetilde{\Omega} = \bigcup_{n=1}^{\infty} \left\{S^{-1}XS : X\in\Omega_n,\ S\in \mathbf{GL}_n\right\}. \tag{1}$$

An *NC domain* is an open and connected NC set. Every operator space E gives rise to the *operator space ball* (or *operator ball*, for short) $\mathbb{B}_E$

$$\mathbb{B}_E = \cup_{n=1}^{\infty}\{X\in M_n(E) : \|X\|_n < 1\}. \tag{2}$$

It will be useful to have a couple of concrete examples at hand. The *row ball* $\mathfrak{B}_d$, which is defined to be $\mathbb{B}_E$ for $E=\mathbb{C}^d$ with the row operator space structure, is given by

$$\mathfrak{B}_d := \mathbb{B}_{(\mathbb{C}^d)_{row}} = \{X\in\mathbb{M}^d : \|X\|_{row} := \|\sum X_jX_j^*\|^{1/2} < 1\}. \tag{3}$$

When $E=\min(\ell_d^\infty)$ is the minimal operator space over ℓ_d^∞ we get the *NC polydisc*

$$\mathfrak{D}_d := \mathbb{B}_{\min(\ell_d^\infty)} = \{X\in\mathbb{M}^d : \|X\|_\infty := \max_{1\le i\le d}\|X_i\| < 1\}.$$

NC analytic functions — definition

A function f from an NC set $\Omega\subseteq\mathbb{M}(E)$ to $\mathbb{M}(F)$ is said to be an *NC function* if: (i) f is graded: $X\in\Omega_n \Rightarrow f(X)\in M_n(F)$; (ii) f respects direct sums: $f(X\oplus Y) = f(X)\oplus f(Y)$; and (iii) f respects similarities: if $X\in\Omega_n$, $S\in GL_n$, and if $S^{-1}XS\in\Omega_n$, then $f(S^{-1}XS) = S^{-1}f(X)S$. For example, *free polynomials* are NC functions in a natural way: if $p(z)=\sum_w c_w z^w$ a free polynomial, then for $A\in\mathbb{M}^d$ one evaluates p on A as follows

$$p(A) = \sum_w c_w A^w,$$

where for a word w in $\{1,2,\ldots,d\}^n$ the monomial $z^w = z_{w_1}\cdots z_{w_n}$ evaluated at A^w is given by $A_{w_1}\cdots A_{w_n}$. We let $\mathbb{C}\langle z\rangle = \mathbb{C}\langle z_1,\ldots,z_d\rangle$ denote the algebra of free polynomials in d noncommuting variables, also referred to as the *free algebra*. A more sophisticated class of NC functions is provided by *NC rational functions* [20, 23, 25, 28, 44]. Remarkably, every bounded NC function is *analytic* and has a Taylor series at every point; in fact, for an NC function in an operator ball, the series around the origin converges in the entire ball [26].

NC varieties and the algebras of interest

Let Ω be an NC domain. For us, an *NC analytic variety* $\mathfrak{V}\subseteq\Omega$ is defined to be the joint zero set of a set of bounded NC functions on Ω. We define $H^\infty(\mathfrak{V})$ to be the algebra of bounded NC functions on $\mathfrak{V}$ equipped with the supremum norm

$$\|f\| = \|f\|_\infty := \sup_{X\in\mathfrak{V}}\|f(X)\|.$$

We define $A(\mathfrak{V})$ to be the closure of polynomials in $H^\infty(\mathfrak{V})$. In particular, when $\Omega=\mathfrak{V}=\mathbb{B}_E$ we have the algebras $A(\mathbb{B}_E)$ and $H^\infty(\mathbb{B}_E)$ at the center of the stage. If $\mathfrak{V}$ is a homogeneous set, then $A(\mathfrak{V})$ equals the set of bounded NC functions that extend to uniformly continuous functions on $\overline{\mathfrak{V}}$ [36]. For notational convenience, we let $\mathscr{A}(\mathfrak{V})$ denote *either* $A(\mathfrak{V})$ *or* $H^\infty(\mathfrak{V})$, and refer to both as *NC function algebras*. It is easy to verify that $\mathscr{A}(\mathfrak{V})$, equipped with the supremum norm and with the pointwise operations of addition and multiplication, satisfies the Blecher–Ruan–Sinclair axioms and so is an operator algebra [11].

2 A pathway to noncommutative function algebras

So, how was I led to study algebras of bounded NC functions on NC varieties? Here is the story.

2.1 From CP-semigroups to subproduct systems to tensor algebras

In my PhD thesis I introduced and developed the notion of a *subproduct system* [38], as a technical tool for analyzing multi-parameter CP-semigroups on von Neumann algebras and their dilations. In their very simplest incarnation, subproduct systems are families $X = \{X_n\}_{n\in\mathbb{N}}$ of Hilbert spaces such that

$$X_{m+n} \subseteq X_m \otimes X_n, \qquad \text{for all } m,n \in \mathbb{N}.$$

The elegant structure invites the pure mathematician to consider subproduct systems and their representations as objects of study in themselves. Every subproduct system corresponds to a homogeneous ideal I_X in the free algebra $\mathbb{C}\langle z\rangle$, and representations of subproduct systems therefore correspond to tuples of operators satisfying certain homogeneous polynomial relations. My PhD advisor Solel pointed out to me that this encapsulates many structures and notions that have been studied in multivariable operator theory, and we were led in [38] to consider the natural shift operators on the X-Fock space $\mathscr{F}_X = \oplus_{n=0}^{\infty} X_m$ and the various operator algebras they generate. For the sake of the story, let me introduce just one: the *tensor algebra* $\mathscr{A}_X$, which is the unital norm closed algebra generated by the shifts. I was drawn to investigate these nonselfadjoint operator algebras along the lines of the pervasive Gelfand-theme from the opening paragraph, and continued investigating them during my postdoc. In [13], with Davidson and Ramsey, we proved the following theorem.

Theorem 2.1 (Davdison–Ramsey–Shalit)**.** *Let X and Y be subproduct systems with finite dimensional fibers. Then, $\mathscr{A}_X$ is isometrically isomorphic to $\mathscr{A}_Y$ if and only if X and Y are isomorphic as subproduct system.*

Later, in [24], with Kakariadis, we solved this problem for completely bounded isomorphism, and conducted a detailed investigation into subproduct systems corresponding to monomial ideals.

2.2 From tensor algebras to commutative algebras of multipliers

Recall that every subproduct systems corresponds to a homogeneous ideal I_X in the free algebra $\mathbb{C}\langle z\rangle$. In the case that I_X contains the commutator ideal $\mathscr{C}$, the X-Fock space $\mathscr{F}_X$ is contained in the symmetric Fock space over X_1 and the X-shifts commute, in which case $\mathscr{A}_X$ is a commutative operator algebra. Every such commutative subproduct system corresponds to a polynomial ideal $J_X = I_X/\mathscr{C}$ in the algebra of commutative polynomials $\mathbb{C}[z] = \mathbb{C}[z_1,\dots,z_d] \cong \mathbb{C}\langle z\rangle/\mathscr{C}$ in d commuting variables.

Let $\mathscr{A}_d$ denote the closure of the shift on symmetric Fock space. Arveson recognized [5] that $\mathscr{A}_d$ can be identified as a closed subalgebra of the multiplier algebra $\mathscr{M}_d := \mathrm{Mult}(H^2_d)$ of a certain reproducing kernel Hilbert space H^2_d on the unit ball $\mathbb{B}_d$, that has come to be known as the Drury–Arveson space and has played a central role in operator theory and function theory (see the survey [18]). The algebra $\mathscr{A}_d$ is the closure of the algebra $\mathbb{C}[z]$ in the multiplier norm. Davidson, Ramsey and I realized that when restricting attention to commutative subproduct systems, we are classifying quotients of $\mathscr{A}_d$ by closed homogeneous ideals: $\mathscr{A}_X \cong \mathscr{A}_d/\overline{J_X}$. These being commutative algebras, we were intrigued by the possibility of classifying them in terms of some natural geometric invariant.

In the case that the ideal J_X is radical, we obtained in [13] very sharp and satisfying results (some difficult technical steps were completed only in the subsequent brilliant contribution by Hartz [16]). Let

$$V_X = V_{\mathbb{B}_d}(J_X) := \{z \in \mathbb{B}_d : p(z) = 0 \text{ for all } p \in J_X\}$$

be the subvariety of the ball corresponding to J_X. The main results of [13, 16] are as follows.

Theorem 2.2 (Davidson–Ramsey–Shalit)**.** *Let X,Y be subproduct systems such that J_X, J_Y are radical. Then, $\mathscr{A}_X$ and $\mathscr{A}_Y$ are isometrically isomorphic iff there is a unitary $U\colon \mathbb{C}^d \to \mathbb{C}^d$ such that $U(V_X) = V_Y$.*

Theorem 2.3 (Davidson–Ramsey–Shalit, Hartz)**.** *Let X,Y be subproduct systems such that J_X, J_Y are radical. Then, $\mathscr{A}_X$ and $\mathscr{A}_Y$ are isomorphic iff V_X and V_Y are biholomorphic, and this happens iff there is a linear bijection $A\colon \mathbb{C}^d \to \mathbb{C}^d$ such that $A(V_X) = V_Y$.*

This has further striking consequences; e.g. if V_X is irreducible or is a nonlinear hypersurface, then if $\mathscr{A}_X$ and $\mathscr{A}_Y$ are isomorphic then they must be isometrically isomorphic, and in fact unitarily equivalent.

2.3 The non radical gap

The paper [13] with Davidson and Ramsey was generally well-received, but I recall one penetrating referee report criticizing us for obtaining the best results only for radical ideals. At the time, I felt that this criticism was somewhat unfair, because I thought that, by Hilbert's Nullstellensatz, we couldn't expect to recover algebra from geometry unless our ideals were radical. The question of how to handle the non radical case kept occupying me, but I wasn't satisfied with the approaches suggested by algebraic geometry. It took me some time to find a solution that aligned with my perspective and mathematical taste.

2.4 The isomorphism problem for complete Pick algebras and the elusive category

Before being able to find the geometric invariant for non radical ideals, Davidson, Ramsey and I moved from tensor algebras, and began studying the isomorphism problem for complete Pick algebras. The class of complete Pick algebras had a small intersection with that of tensor algebras, but contained a rich variety of algebras that generalize in a completely different direction from what we studied earlier.

The setting is as follows: let H^2_d be the Drury–Arveson space, the RKHS on the unit ball $\mathbb{B}_d \subset \mathbb{C}^d$ with reproducing kernel

$$k(x,y) = (1 - \langle x,y\rangle)^{-1},$$

and let $\mathscr{M}_d = \mathrm{Mult}(H^2_d)$ be its multiplier algebra. Let $V \subseteq \mathbb{B}_d$ be a *variety*, by which we mean the joint zero set of a family of multipliers. Let

$$H_V = H^2_d\big|_V \cong \overline{\mathrm{span}\{k_v : v \in V\}},$$

and put $\mathscr{M}_V = \mathrm{Mult}(H_V)$. It is well known that $\mathscr{M}_d$ has the complete Pick property [2], and it follows that $\mathscr{M}_V = \mathscr{M}_d\big|_V$. By the Agler–McCarthy theorem [1], every irreducible multiplier algebra with the complete Pick property is of this form.

In [14], Davidson, Ramsey and I proved the that $\mathscr{M}_V \cong \mathscr{M}_W$ completely isometrically if and only if $V \cong W$ via an automorphism of $\mathrm{Aut}\,\mathbb{B}_d$. The homogeneous case suggested that $\mathscr{M}_V \cong \mathscr{M}_W$ algebraically is equivalent to $V \cong W$ biholomorphically. In [27] with Kerr and McCarthy, we showed that if V and W are one-dimensional subvarieties of the ball with no singularities or self-intersections on the boundary, then $\mathscr{M}_V \cong \mathscr{M}_W$ algebraically if and only V and W are biholomorphic. It started to seem as though the category of subvarieties with biholomorphisms is equivalent to the category of complete Pick algebras with algebraic isomorphisms. However with Hartz and Davidson we showed this fails already for analytic discs with a single self-intersection on the boundary [12].

In [14], we showed that if V and W are a finite union of irreducible or discrete varieties then $\mathscr{M}_V \cong \mathscr{M}_W$ as algebras implies that $V \cong W$ via a *multiplier biholomorphism*; i.e., the component functions of the biholomorphism are not merely analytic, they are elements of the multiplier algebra $\mathscr{M}_d$. However, in general, multiplier biholomorphism does not imply that the algebras are isomorphic [12]. The reason: *multiplier biholomorphism is not an equivalence relation!* Since algebraic isomorphism is an equivalence relation for multiplier algebras, this raised the question whether the variety V is a complete invariant for the algebra $\mathscr{M}_V$; if it is, then what are the morphisms of the corresponding category; and if it isn't, then what is the correct invariant?

2.5 NC varieties — the missing link

Around 2015 I realized that all the operator algebras that I have been working on can be realized as algebras $\mathscr{A}(\mathfrak{V})$ of bounded NC functions on NC subvarieties of the row ball $\mathfrak{B}_d$. These connections (and many more) were sorted out by Salomon, Shamovich and myself in the two landmark papers [34] and [35]. For example, if X is a product system and I_X the corresponding ideal in $\mathbb{C}\langle z\rangle$, then $\mathscr{A}_X = A(\mathfrak{V})$ — the algebra of uniformly continuous NC functions on $\mathfrak{V}$, where $\mathfrak{V}$ is the NC variety

$$\mathfrak{V} = V_{\mathfrak{B}_d}(I_X) := \{X \in \mathfrak{B}_d : p(X) = 0 \text{ for all } p \in I_X\}.$$

In [34] we obtained the following strengthened version of Theorem 2.1.

Theorem 2.4 (Salomon–Shalit–Shamovich)**.** *If $\mathfrak{V},\mathfrak{W} \subseteq \mathfrak{B}_d$ are homogeneous varieties in the row ball, then $A(\mathfrak{V})$ is isometrically isomorphic to $A(\mathfrak{W})$ if and only if there is unitary $U : \mathbb{C}^d \to \mathbb{C}^d$ such that $U(\mathfrak{V}) = \mathfrak{W}$. In fact, every isometric isomorphism $\varphi : A(\mathfrak{V}) \to A(\mathfrak{W})$ is implemented by a composition by a free automorphism $\alpha \in \mathrm{Aut}\,\mathfrak{B}_d$, in the sense that*

$$\varphi(f) = f \circ \alpha, \quad \textit{for all } f \in A(\mathfrak{V}).$$

Note that this theorem is stronger than Theorem 2.1, because it not only gives conditions for the existence of an isometric isomorphism, but it also describes how every isometric isomorphism is given. This gives a categorical equivalence between the algebras $A(\mathfrak{V})$, $\mathfrak{V} \subseteq \mathfrak{B}_d$ (equivalently, the category of tensor algebras $\mathscr{A}_X$ with $\dim X_1 = d$) with completely isometric isomorphisms, and the category of homogeneous subvarieties of $\mathfrak{B}_d$ with restrictions of automorphisms of the ambient $\mathfrak{B}_d$ as morphisms. These improvements are due to the passage to an NC function algebraic setting, revealing the correct geometric invariant.

In particular, specializing Theorem 2.4 to commutative subproduct systems, we recover Theorem 2.2 as well as its generalization to non radical ideals, thereby closing part of the non radical gap.

In a similar fashion, we find a candidate for the classifying invariant for complete Pick algebras. First, we proved that the multiplier algebra $\mathscr{M}_d = \mathrm{Mult}(H^2_d)$ can be identified as

$$\mathscr{M}_d \cong H^\infty(\mathfrak{CB}_d),$$

where

$$\mathfrak{CB}_d = V_{\mathfrak{B}_d}(\mathscr{C}) = \{X \in \mathfrak{B}_d : X_iX_j = X_jX_i \text{ for all } i,j = 1,\dots,d\}$$

is the *free commutative ball*. We then have the completely isometric identification $\mathscr{M}_V \cong H^\infty(\mathfrak{V})$ with the algebra of bounded NC functions on the variety $\mathfrak{V}$, which a sort of *NC Zariski closure* of V

$$\mathfrak{V} = V_{\mathfrak{CB}_d}(I_{\mathscr{M}_d}(V)),$$

where

$$I_{\mathscr{M}_d}(V) := \{f \in \mathscr{M}_d : f(z) = 0 \text{ for all } z \in V\}$$

is the annihilator of V in $\mathscr{M}_d$. The NC varieties play the role of complete invariants of the algebras $\mathscr{A}(\mathfrak{V})$ up to completely isometric isomorphism [34]. However, *they must fail* to serve as complete invariants for the algebras up to completely bounded isomorphism — the correct geometric invariant is the similarity envelope of the NC variety, as we showed in [35].

2.6 Bonus: the perfect Nullstellensatz

We have seen above that noncommutative varieties can play the part of a geometric invariant, even in the case when the algebras under consideration are commutative. A heuristic explanation for why this happens is provided by the following *perfect Nullstellensatz*, which Salomon, Shamovich and I rediscovered [34] (this result follows from a more general old result of Eisenbud–Hochster [15]).

Let $\mathfrak{C}\mathbb{M}^d$ be the NC set consisting of all d-tuples of commuting $n \times n$ matrices, for all n. Put

$$V_{\mathfrak{C}\mathbb{M}^d}(J) = \{X \in \mathfrak{C}\mathbb{M}^d : p(X) = 0 \text{ for all } p \in J\},$$

and

$$I_{\mathbb{C}[z]}(V_{\mathfrak{C}\mathbb{M}^d}(J)) = \{p \in \mathbb{C}[z] : p(X) = 0 \text{ for all } X \in V_{\mathfrak{C}\mathbb{M}^d}(J)\}.$$

Theorem 2.5 (The perfect Nullstellensatz)**.** *Let J be a polynomial ideal in $\mathbb{C}[z] = \mathbb{C}[z_1,\dots,z_d]$. Then*

$$I_{\mathbb{C}[z]}(V_{\mathfrak{C}\mathbb{M}^d}(J)) = J.$$

Classical varieties consisting of scalar zeros can be identified with character spaces, that is, with spaces of one-dimensional representations, and can never give us more than Hilbert's Nullstellensatz:

$$I_{\mathbb{C}[z]}(V_{\mathbb{C}^d}(J)) = \sqrt{J}.$$

In order to deal with multiplicity of zeros, we must work with the NC variety $V_{\mathfrak{C}\mathbb{M}^d}(J)$ consisting of the matrix zeros of the ideal J. The NC variety corresponds to finite dimensional representation of $\mathbb{C}[z]/J$, and can deal with multiplicity once the dimensions are large enough. It is worth pointing out that if we took the variety consisting of “all" the zeros of the ideal J (i.e. including infinite dimensional representations) then the theorem would be a tautology provable by a couple of lines of abstract nonsense, on the one hand, and useless on the other, because how does one get a handle on “all" these zeros? The NC variety provides just the right amount of noncommutativity to serve as a nontrivial yet tangible invariant, which is strong enough to recover J.

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