The universal commuting dilation constant

Orr Shalit

based on works with Malte Gerhold, Satish Pandey, Marcel Scherer and Baruch Solel

and on recent joint work by Ian Thompson

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Dilations of tuples $A \prec B$

$A = (A_1, \dots, A_d) \in B(\mathcal{H})^d$
$B = (B_1, \dots, B_d) \in B(\mathcal{K})^d$, where $\mathcal{K} \supset \mathcal{H}$

Definition

A is said to be a **compression** of B if

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We then say that B is a **dilation** of A, and we write $A \prec B$.

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New definition: $A \prec B$ if $A' \prec B'$ for $A \sim A'$ and $B \sim B'$.

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3. For $d = 2$: $\sqrt{2} < 1.543 \leq C_2 \leq 2 = \sqrt{2 \cdot 2} = 2$

Dilation constants

For tuples $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$ of unitaries we define

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How to improve bound from below? Estimate the dilation constant $c(u, u_0)$ where is u is some very noncommutative tuple of unitaries.

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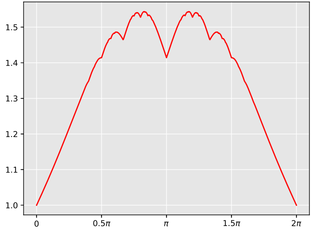

This pushes up lower bound: $C_2 \geq \max_\theta c_\theta \geq 1.543 > \sqrt{2}$

Free Haar unitaries

Example: The free Haar unitaries $u^\lambda = (u_1^\lambda, \ldots, u_d^\lambda)$ on $\ell^2(\mathbb{F}_d)$, where $\mathbb{F}_d = \langle g_1, \ldots, g_d \rangle$ is the free group and we use the left regular rep'n:

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It is known that $C_d \geq \sqrt{d}$ . . . this is a lousy lower bound.
Shows the difference between $C^*(\mathbb{F}_d)$ and $C_r^*(\mathbb{F}_d)$!

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The free Haar unitaries are not very noncommutative

Upper-bounding C_d via the free Haar unitaries u^λ

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Recover the nontrivial:

$$C_d = c(\mathbf{u}, u_0) \le \frac{d}{\sqrt{2d-1}} \times 2\sqrt{1-\frac{1}{2d}} = \sqrt{2d}$$

but for $d = 2$ we get the trivial $C_2 \le 2$.

Upper-bounding C_2 via random unitary matrices

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Let $U^N = (U_1^N, U_2^N)$ be a sequence of pairs of independent Haar distributed unitary $N \times N$ matrices. Then

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But for fixed N the constant $c(U^N, u_0)$ can be estimated numerically.

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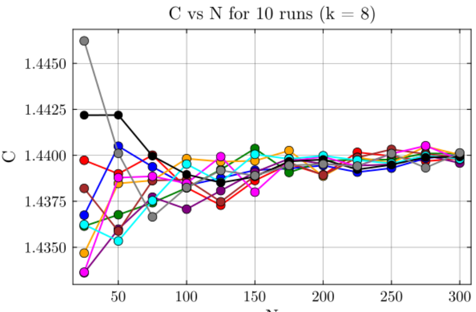

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$\implies$ $\widetilde{u}, \widetilde{v}$ are commuting unitaries:

$$\widetilde{u}\widetilde{v} = \sum u^{k+1} v u^{-k} \otimes S p_k = \sum u^{k+1} v u^{-k} \otimes p_{k+1} S = \widetilde{v}\widetilde{u}$$

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1. *$C_2 \leq 2/\sqrt{\varphi} \approx 1.573$ i.e. every u, v dilate to $2/\sqrt{\varphi}\times$commuting unis.*
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Proof attempt: $S\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, $Se_k = e_{k+1}$. Let $p_k\colon \ell^2(\mathbb{Z}) \to \mathbb{C}e_k$

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Put $\xi_N := \frac{1}{\sqrt{2N+1}} \sum_{k=-N}^{N} e_k$ and $\iota := \mathrm{id} \otimes \xi_N\colon \mathcal{H} \to \mathcal{H} \otimes \ell^2(\mathbb{Z})$

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Thank you!

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